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THE FLOW OF WATER IN TRANSITION SECTIONS OF RECTANGULAR OPEN CHANNELS AT SUPERCRITICAL VELOCITIES

by

Warren E. Wilson

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy, in the Department of Mechanics and Hydraulics, in the Graduate College of the State University of Iowa

August 1940

This thesis is hereby approved as a creditable report on an engineering project or research carried out and presented in a manner which warrants its acceptance as a prerequisite for the degree for which it is submitted. It is to be understood, however, that neither the Department of Mechanics and Hydraulics nor the thesis adviser is responsible for the statements made of for the opinions expressed.

Thesis Adviser

Acting Chairman of Department

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To Professor E. W. Lane, associate director in charge of the Institute of Hydraulic Research and professor of hydraulic engineering of the State University of Iowa, the author is indebted for the suggestion of the study undertaken herein, and the assistance rendered in carrying it out.

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I. INTRODUCTION

In an effort to develop a method to be used in designing transition sections for rectangular open channels in which the flow would be at supercritical velocities the study described herein was made. It was realized that this problem is closely allied to that of supercritical flow in curved open channels, hence due consideration was given to the results obtained by Ippen and Knapp at the California Institute of Technology in the investigation of such flows.

References in current periodical literature to some unusual and obviously incompletely understood phenomena associated with flow in transitions in open channels served as an incentive to carry on the work. It was at first believed that in addition to an experimental investigation a complete theoretical analysis might well be possible. However, as the work progressed a number of revisions, including a considerable narrowing of the scope of the investigation, were found imperative.

The original plan envisaged the development of design procedures for various types of transitions. A transition from a rectangular channel of a given width to one of greater width, effected by means of walls forming reversed curves tangent at either end to straight parallel channel walls, was the first to be studied. It was soon apparent that this type of transition involved a flow pattern far too complicated to permit complete analysis at the present time. A transition involving simple curved walls tangent at either end to straight walls which are initially parallel and which diverge downstream from the curved section was found to be the most complex structure the design of which could be undertaken with some degree of confidence.

The experimental work involved a large number of experiments on transitions with reversal of wall curvature as well as with the simpler diverging walls. A wide range of pertinent ratios of lengths involved in the geometry of the transitions as well as a wide range of Froude numbers, the latter being limited by the velocities obtainable in the models, were used to facilitate the formulation of an empirical solution in the event that no analytical one was found.

A theoretical analysis of the flow bounded by a diverging curved wall with non-hydrostatic pressure distribution with particular reference to pressures less than hydrostatic was made and an interpretation of the experimental data in the light of this analysis is given. Equations were derived which aid in making a qualitative study of the experimental data, but which are not as yet sufficiently complete to predict quantitatively the elements of the flow at a diverging curved wall or in the main body of the liquid in the channel. The limitations upon the use of the data, which were obtained with relatively small structures, and similar data on small models for predicting the elements of the flow in a full scale channel are set forth.

II. PROPAGATION OF DISTURBANCES IN SUPERCRITICAL FLOW

(a) Wave velocity with hydrostatic pressure distribution

When water flows in an open channel at a velocity in excess of the velocity of propagation of a small wave in water of the given depth, the flow is said to be at supercritical velocity, the critical velocity being equal to the elementary wave velocity. Small disturbances such as increments or decrements of the water depth are propagated at wave velocity and cannot proceed upstream in supercritical flow. The value of the elementary wave celerity c relative to the fluid is given by the expression,

$$c = \sqrt{gd} \tag{1}$$

in which g is the acceleration due to gravity and d is the depth of the water. The derivation of this equation and all those described herein may be found in the appendix. It is assumed that the wave height is small and the wave length great relative to depth and that the pressure distribution is hydrostatic.

(b) Wave angle

If a small disturbance is continuous in nature, such as would be the result of continuous flow past a change in the direction of a channel wall, a wave front is formed, starting at the disturbance point and extending into the flowing water at an angle with the original direction of flow usually termed β , the wave angle. This angle is defined by the expression

$$\sin \beta = \frac{c}{u} \qquad (2)$$

in which c is the wave velocity defined by Equation (1) and u is the mean velocity in the original direction of flow. Upon substitution of the expression for c we have,

$$\sin \beta = \left(\frac{gd}{u^2}\right)^{\frac{1}{2}} \qquad (3)$$

or

$$\sin \beta = \frac{1}{F^{\frac{1}{2}}} \qquad (3a)$$

where F is the Froude number defined by the expression $F = \frac{u^2}{gd}$.

(c) Depth change on passing under a wave front

As the water passes under a wave front, the component of momentum perpendicular to the wave front undergoes a change in magnitude proportional to the negative change in depth of the water, whereas the component parallel to the wave front remains unchanged, since the elevation of the surface along the wave front is constant. The changes in depth and components of velocity are represented in Fig. 1.

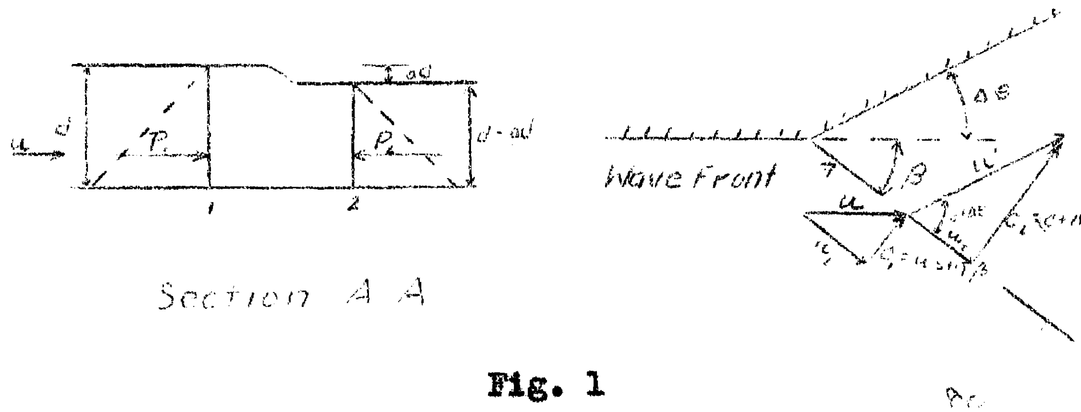

Fig. 1

A simple derivation on the basis of continuity, geometry of the vector diagram, and the momentum principle leads to the following equation for the increment of depth in terms of angular change in the direction of the boundary, wave angle and velocity of the water,

$$\Delta d = \frac{u^2}{g} \tan\beta \cdot \Delta\theta \tag{4}$$

If the pressure distribution in the body of the liquid may be considered essentially hydrostatic and if friction losses may be ignored, the flow along a curved boundary may be successfully analyzed on the basis of the elementary principles set forth above. A most successful analysis of this type was made by Ippen and Knapp in "A Study of High Velocity Flow in Curved Sections of Open Channels", Pasadena, California, March 29, 1936.

In that study an equation derived on the assumption of constant velocity along the wall proved best in describing the wall profile of the water surface along the curved wall. This equation will be used herein for purposes of comparison and is given here,

$$\frac{d}{d_o} = F_o \sin^2\left(\beta_o \pm \frac{\theta}{2}\right) \tag{5}$$

in which the subscript zero refers, as it will throughout this discussion, to conditions at the beginning of the curved wall. The significance of the terms in Equation (5) is as follows:

d = depth of the water

β = the wave angle

θ = the total angle between the tangent at the beginning of curve and the tangent at the point at which the depth is d.

F = the Froude number defined by the expression

$$F = \frac{u^2}{gd}$$

(d) Wave velocity with non-hydrostatic pressure distribution

Let it be assumed that the pressure distribution at the wall and within the body of the liquid is not hydrostatic. In Fig. 2 is shown a section similar to section A-A of Fig. 1 taken perpendicular to a wave front, which will now be defined as a line along which there is no change of momentum of the liquid.

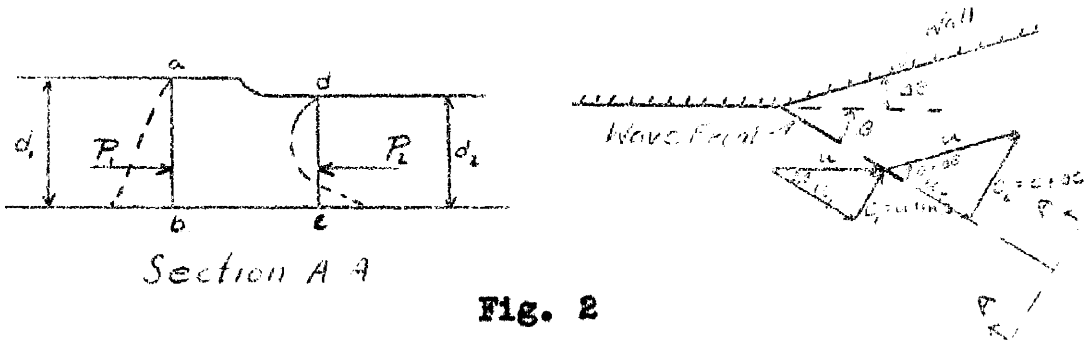

Fig. 2

The forces P_1 and P_2 per unit width acting on the ends of the free body are the non-hydrostatic pressure forces which will be represented thus,

$$P_1 = k_1 \rho \frac{gd^2}{2} \quad \text{and} \quad P_2 = k_2 \rho \frac{gd^2}{2} \tag{6}$$

in which ρ is the unit density of the liquid and the coefficients k_1 and k_2 represent the factors by which the total hydrostatic pressure corresponding to the depths d_1 and d_2 must be multiplied to obtain the actual total pressure. It will be assumed that the friction forces acting on the element of liquid are negligible in comparison with the pressure forces.

We may now proceed to a derivation of the velocity of propagation of a small wave with conditions as described above. The velocity is assumed to be uniformly distributed throughout a vertical section. The momentum equation is written in the following form,

$$p_1 - p_2 = \rho \left(c_2^2 d_2 - c_1^2 d_1 \right) \tag{7}$$

From the continuity relationship we have,

$$c_1 d_1 = c_2 d_2 \tag{8}$$

Assume now that the wave height is small, that is $d_1 - d_2$ is a very small quantity compared to d_1. The simultaneous solution of Equations (6), (7), and (8) leads to the following equation for the wave velocity c,

$$c = \sqrt{gd\left[k_1 + \frac{k_1 d}{2(d_1 - d_2)}\left(1 - \frac{k_2}{k_1}\right)\right]} \tag{9}$$

For the case of hydrostatic pressure distribution $k_1 = k_2 = 1$. In this case Equation (8) reduces to Equation (1) which gives the velocity of propagation of the elementary wave with hydrostatic pressure distribution.

(e) Wave angle

By definition the wave angle is given as the ratio of the wave velocity to the mean velocity of the liquid. Equation (2) thus gives the sine of the wave angle in the general form. Equation (3) gives the value for the particular case of hydrostatic pressure distribution. The following equation gives the sine of the wave angle for the case of non-hydrostatic pressure distribution,

$$\sin\beta = \sqrt{\frac{gd}{u^2}\left[k_1 + \frac{k_1 d}{2(d_1 - d_2)}\left(1 - \frac{k_2}{k_1}\right)\right.} \qquad (10)$$

which in turn reduces to the equation for the wave angle with hydrostatic pressure distribution when $k = 1$.

(f) Depth change on passing under a wave front

Employing the momentum equation and with regard for the geometry of the vector diagram of Fig. 2, the following equation for the depth change experienced by the liquid passing under a wave front may be derived,

$$\Delta d = \frac{u^2}{k_2 g} \tan\beta \cdot \Delta\theta + \frac{d}{2}\left(1 - \frac{k_1}{k_2}\right) \qquad (11)$$

which may be reduced to the more significant form

$$\frac{\Delta d}{\Delta\theta} = \frac{d}{4k}\left[-\frac{\Delta k}{\Delta\theta} + \sqrt{\left(\frac{\Delta k}{\Delta\theta}\right)^2 + 16\,k\,F}\right] \qquad (12)$$

through the assumption, which is valid for large Froude numbers, that $\sin\beta = \tan\beta$. For the case of hydrostatic pressure distribution $k = 1$ and $\Delta k/\Delta\theta = 0$ and Equation (11) will reduce to Equation (4) which was derived specifically for the case of hydrostatic pressure distribution. In using Δk to represent $k_1 - k_2$ it is apparent that the sign of $\Delta k/\Delta\theta$ is positive as is the sign of $\Delta d/\Delta\theta$ when k and d respectively increase in the direction of propagation of the wave indicated in

Fig. 2.

(g) Interpretation of equations

Equation (12) may be employed to make a very significant comparison between the rate of change in the depth $\Delta d/\Delta\theta$ which may be expected with non-hydrostatic pressure and the corresponding rate of change of depth if the pressure distribution were hydrostatic. Throughout the following discussion, for the sake of brevity the term "normal" will be used to refer to elements of flow with hydrostatic pressure distribution, and the subscript n will denote the quantities which are elements of the normal flow.

If Equation (4) is solved for $\Delta d/\Delta\theta$ and if $\tan\beta$ and u and g are evaluated in terms of the Froude number there results:

$$\frac{\Delta d}{\Delta\theta} = F\,\frac{d}{(F-1)^{\frac{1}{2}}} \tag{13}$$

If large values of F are to be considered the simplification of dropping the factor unity in the denominator may be made and the following simple expression results:

$$\frac{\Delta d}{\Delta\theta} = d\cdot F^{\frac{1}{2}} \tag{14}$$

The ratio of the expected angular depth gradient $\Delta d/\Delta\theta$ to the normal gradient may now be expressed in the form,

$$\frac{\Delta d/\Delta\theta}{(\Delta d/\Delta\theta)_n} = \frac{\frac{d}{4k}\left[-\frac{\Delta k}{\Delta\theta} + \sqrt{\left(\frac{\Delta k}{\Delta\theta}\right)^2 + 16Fk}\;\right]}{d\cdot\sqrt{F}} \tag{15}$$

A criterion may now be established which will indicate for any given set of conditions whether the angular depth gradient $\Delta d/\Delta\theta$ may be expected to be greater or less than the normal value. If the ratio expressed in Equation (15) has the value unity the angular depth gradient with non-hydrostatic pressure distribution will be equal to the normal value. Setting the ratio equal to unity and reducing it to simplest form one obtains for the criterion the expression,

$$\frac{\Delta k}{\Delta\theta}\ \frac{1}{1 - k} = 2\ F^{\frac{1}{2}} \qquad (16)$$

It is now apparent that for any given local value of the Froude number there is a certain combination of the factors describing the characteristics of the flow which will result in an angular depth gradient equal to that for normal flow. However, if the characteristics of the flow are such that the combination of terms on the left hand side of Equation (16) is greater or less than the quantity $2\ F^{\frac{1}{2}}$ the surface slope will be less or greater than normal respectively. That such is the case is apparent from Fig. 3 wherein are plotted values of the angular depth gradient ratio for various assumed combinations of the variables F, k and $\Delta k/\Delta\theta$. For any given values of F and k an increase or decrease in $\Delta k/\Delta\theta$ will result in a decrease or increase in the angular depth gradient ratio respectively, which demonstrates the above statement.

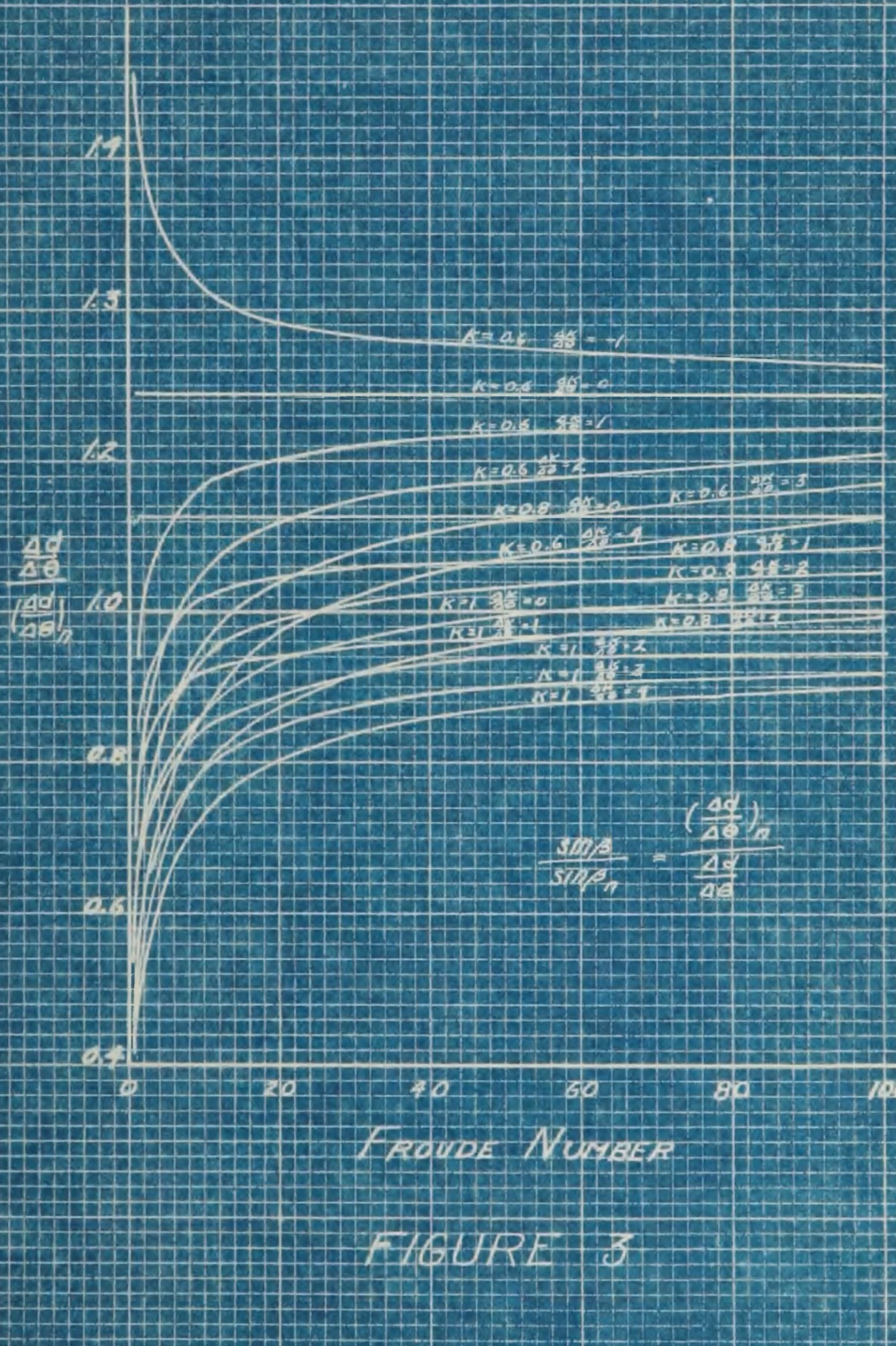
1.4
1.3
1.2
1.0
0.8
0.6
0.4
0
20
40
60
80
100
FROUDE NUMBER
$\frac{\sin\beta}{\sin\beta_n} = \frac{\left(\frac{\Delta d}{\Delta\theta}\right)_n}{\frac{\Delta d}{\Delta\theta}}$

FIGURE 3

The same criterion will indicate the relative value of the wave angle. However in this case values of the quantity on the left of Equation (16) greater or less than $2\sqrt{F}$ indicate values of β greater or less than normal respectively. Obviously the same criterion applies to the wave velocity since the wave velocity divided by the average velocity of the flow is equal to $\sin\beta$. In this connection an interesting form of the equation for the angle β is obtained by a transformation similar to that used in obtaining Equation (12). This equation is,

$$\sin\beta = \left(\frac{k}{F}\right)^{\frac{1}{2}} \left[1 + \frac{2}{-1 + \left(1 + 16Fk\left(\frac{\Delta\theta}{\Delta k}\right)^2\right)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \qquad (17)$$

These equations may be used to advantage in discussing the flow at a curved boundary since it has been established experimentally that the pressure at the wall is in certain cases non-hydrostatic. It is at present impossible to integrate the equation for $\Delta d/\Delta\theta$ to obtain the depth directly as a function of the variables F and k since these must first be determined experimentally or a method must be devised to relate the factor k directly to the curvature of the surface, the Froude number and the depth. The interdependence of these quantities is apparent from the fact that the subnormal pressure results from vertical acceleration, which is a function of the surface curvature, which in turn depends upon the Froude number and relative curvature of the wall. A method for the determination of the surface contours in supercritical flow based on the analogy with supersonic gas flows has been devised and is admirably set forth by Ernst Preiswerk in a work which has been translated and published by the National Advisory Committee for Aeronautics in Technical Memorandums #934 and 935. In a series of experiments carried on at the Institute for Aerodynamics of the Eidgenoessische Technische Hochschule, Zurich, the validity of this method was demonstrated. However, it is applicable only when the vertical components of the acceleration of the fluid are negligible. Obviously such a method cannot be used satisfactorily for the analysis of flows discussed herein. A similar method of attack would be desirable for that type of flow, but the formulation of such an analysis is beyond the scope of the present work which seeks only to point out the existence of the problem and the general characteristics of the flow.

(h) Limitations

In order to interpret the flow properly in terms of the equations set forth above, the following factors must be kept in mind. In deriving the equations a wave front was defined as a line along which there is no change in momentum of the liquid. A wave front is therefore no longer fully characterized by the feature that all points on it have the same surface elevation, as was the case for the elementary wave theory. Rather the wave fronts are lines in the direction of which there is no net accelerating force acting on the liquid. Surface elevation contours will not serve now as wave fronts since the true wave front is also determined in part by contours of constant mean pressure. However, the characteristics of the surface contours do to some extent indicate the shape of the wave fronts, and are useful in studying certain features of the flow. The wave fronts are determined by the fact that the quantity kd^2 must be a constant along these lines. This follows from the fact that there is no net force in a given direction when $P_1 - P_2$ is zero. From the definition of k the necessity for the constant value of kd^2 follows immediately.

It is to be noted that the increments Δd and Δk are defined as the increments in d and k respectively perpendicular to the wave front. The increment $\Delta\theta$ in the ratios $\Delta d/\Delta\theta$ and $\Delta k/\Delta\theta$ indicates the change in direction of the velocity vector upon passing under the wave front. The directional character of these ratios is important since the pressure gradient varies with the direction in which it is measured.

In order to evaluate the factor k from the experimental data it is necessary to determine the unit pressure at various points on a vertical line extending from surface to bottom of liquid. This would make possible a computation of the total pressure on a vertical area. By definition k is the ratio of this total pressure to the hydrostatic pressure on an equal area similarly located.

(i) Summary

To summarize the principle features of the extended theory of the propagation of small disturbances in supercritical flow the following statements may be made.

1. The wave velocity and wave angle as well as the angular depth gradient may be greater or less than the normal values depending upon the relative local values of the quantities k and $\Delta k/\Delta\theta$ for a given Froude number.

2. If the angle β is greater than normal the angular depth gradient will be less and if the angle is less than normal the angular depth gradient will be greater than normal.

3. In order that no disturbance may be propagated it is not only necessary that the local value of k be zero but also that $\Delta k/\Delta\theta$ be zero. This is apparent from a study of Equation (9) for the wave velocity.

4. Wave fronts do not coincide with the surface elevation contours of the body of the liquid in the channel. The determination of the location and form of the wave fronts requires a knowledge of the pressure distribution within the fluid.

5. Prediction of the elements of the flow including the surface profile at the curved wall in any but

an empirical manner must await further analysis of the mechanics of the flow with special reference to the evaluation of the pressure in terms of the surface slope, the relationship being dependent upon the components of the acceleration within the body of the liquid.

(j) Effect of friction

If a viscous liquid flows at supercritical velocity on a horizontal surface in a channel of uniform width the loss in mechanical energy due to conversion into heat by the viscous forces will result in an increase in depth of the liquid. This may be verified either by a direct consideration of the specific energy diagram or by an inspection of the differential equation of varied flow. This equation in general form is,

$$\frac{dd}{dx} = \frac{s_o - \frac{u^2}{c^2R}}{1 - \frac{u^2}{gd}}$$

where R is the hydraulic radius, s_o is the bottom slope, c is the Chezy coefficient, x is the coordinate in the direction of which the velocity u is taken. When the velocity is supercritical the term u^2/gd which is the Froude number is large compared to unity. Also for the case of flow on a horizontal surface s_o is zero. The equation then reduces to,

$$\frac{dd}{dx} = \frac{-\frac{u^2}{c^2R}}{1 - \frac{u^2}{gd}}$$

This quantity is obviously positive indicating an increase in depth. This contrasts with the decreasing depth of subcritical flow. It may be concluded therefore that the effect of friction on the depth would be to increase it above the value which would be expected for a perfect fluid.

(k) Description of flow in transitions

Since transitions are combinations of curved and straight channel walls, a good idea may be formed of the nature of the flow which may be expected on the basis of the principles set forth above. In Figure 4 are shown two simple types of transitions. Consider first, to eliminate complicating factors, a transition of the type shown in Fig. 4a, of very great width. As the water enters the diverging section, the surface at the wall will fall in accordance with the elementary theory and Equation (5) will adequately describe the surface profile at the wall providing the assumptions made in the derivation are substantially fulfilled, i.e., vertical acceleration is negligible and friction losses may be ignored. In order that such be the case, the curvature must be slow enough to insure hydrostatic pressure distribution and yet not so slow that the friction effects are appreciable. The surface contours will then be straight lines making the angle β with the wall at all points. This angle β will decrease in magnitude along the wall as the depth decreases since observations indicate that the velocity remains essentially constant.

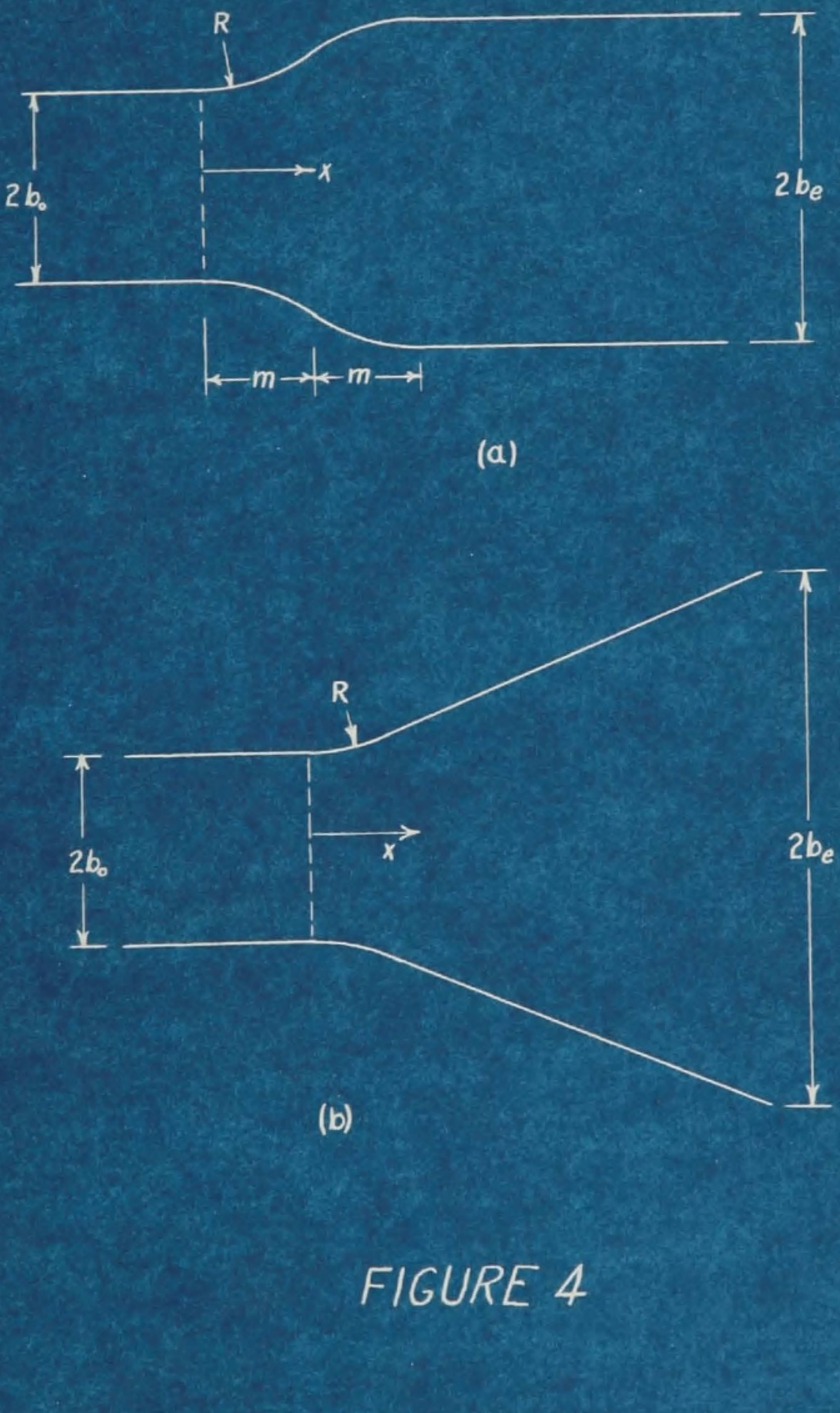

FIGURE 4

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In Fig. 5 is shown a sketch of the flow along a diverging boundary which for the sake of clarity in description is shown as a series of short straight segments each making the angle $\Delta\theta$ with the preceding segment. The wave fronts then appear as lines with finite spacing. The streamlines will be as indicated in the sketch. The spacing of the streamlines gradually becomes greater as the flow proceeds along the boundary. As the water passes under each successive wave front it changes in direction so that the flow is always parallel to the corresponding segment of the wall. In the case of a smooth curve the flow would be modified to the extent that the wave fronts would be spaced at infinitesimal distances, and the streamlines would be smooth curves.

In Fig. 6 is shown a transition of a slightly more complicated type. The flow up to the point of inflection of the curve will be as previously described. After passing the point of inflection of the curve the effect will be similar to that in the initial part of the channel but with increasing rather than decreasing depth and with the streamlines becoming closer spaced rather than wider spaced as before. The ultimate depth attained at the straight wall will be identical with the initial depth and the spacing of the streamlines will assume the original value. This restoration of initial conditions will occur providing there is no influence of another boundary in this

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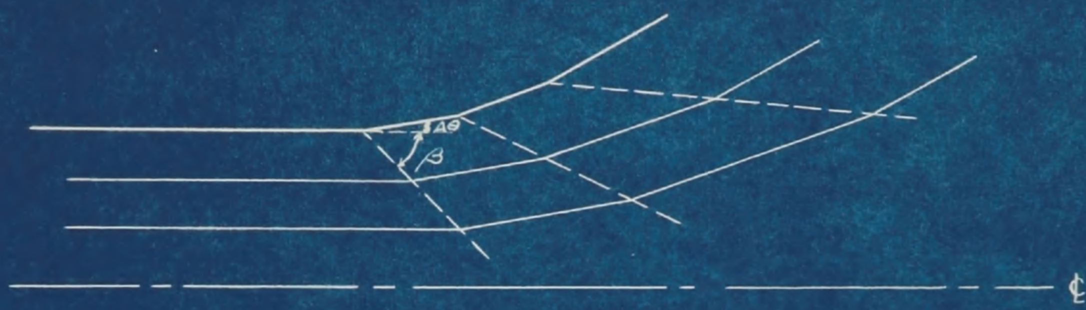


FIGURE 5

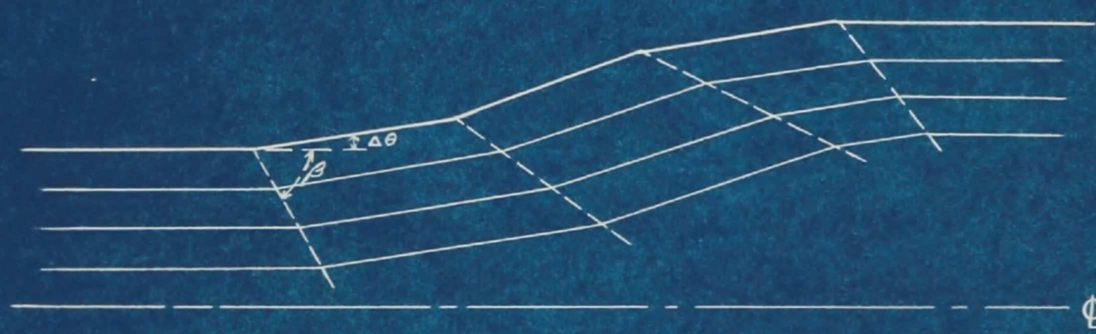


FIGURE 6

region, the boundary drag is negligible and the waves are of the same type for both convex and concave curvature of the wall. If these conditions are fulfilled the liquid will be moving in the same direction with undiminished velocity after passing the curved wall. There is therefore no net change in momentum, and consequently no change in depth.

If a channel of the first type is of finite width, waves crossing from the opposite sides of the channel will augment the depth change at all points of intersection, in accord with the principles of wave interference. The surface contours, moreover, will then be curved lines since interference of waves augmenting the depth change will produce lesser depths in the central region of the flow in the channel along a given wave front than exist at the wall. It is therefore necessary to go upstream to find a point with a depth equal to that at the wall on a given wave front. The angle between contours and wall will then equal the wave angle only at points unaffected by waves crossing the channel. Similarly in a transition of the second type the negative waves crossing the channel will reduce the effect of the positive wave. The surface contours will again be curved lines.

If the friction losses may not be neglected, their effect will be evidenced at the wall by an increase in the elevation of the water surface and a corresponding distortion of the wave fronts.

If the pressure distribution is not hydrostatic, the effects indicated in the development of the equations of wave velocity, wave angle, and change in depth for non-hydrostatic pressure distribution may be expected. Proceeding again from the simple to the complex, the flow at the diverging curved wall may first be discussed. As the liquid enters the diverging section the deviation of the wall from a straight line produces a reduction in the wall pressure, which in turn produces a vertical acceleration of the liquid. This pressure reduction has an immediate effect upon the wave velocity, wave angle, and slope of the liquid surface, in that these quantities will be either larger or smaller than the normal values depending upon the relative values of the factor k and the ratio $\Delta k/\Delta\theta$ in relation to the local Froude number. A quantitative evaluation of k in terms of the surface slope and Froude number has not yet been made, hence only trends may be indicated.

The relative radius and Froude number together determine the magnitude of the pressure reduction. For short relative radii, or large Froude number, the vertical accelerations of the particles are large; consequently a considerable reduction in pressure is to be expected. Fig. 3 is a graphical representation of the equation for the angular depth gradient from which may be obtained definite information on the general form of the surface curve. If the liquid is assumed to enter the channel with uniform velocity throughout the cross section, hydrostatic pressure distribution, and with negligible boundary drag the values of k and the ratio $\Delta k/\Delta\theta$ would be unity and zero respectively. The point in Fig. 3 indicative of this initial condition would lie on the horizontal line indicating the value of ratio $\frac{\Delta d}{\Delta\theta}/(\frac{\Delta d}{\Delta\theta})_n$ equal to unity. Its position on this line would be determined by the initial Froude number.

The use of the diagram to predict surface profiles is not feasible since there is one degree of indeterminancy involved due to the fact that k has not yet been expressed as a function of F and the relative radius. The diagram will be used in the analysis of flows of this type for which data are available. The streamline diagram and surface contour diagram for a flow of the type under discussion would constitute a modification of those shown in Fig. 5 for the case of hydrostatic pressure distribution. The chief differences would be curved rather than straight wave fronts, and different angles between contours and wall and wave fronts and wall.

The effects of the non-hydrostatic pressure distribution on the flow in a transition of the second type would center primarily in the upstream portion of the transition. However, secondary effects would be evident in such a transition of finite width since the waves crossing the channel would have different angles than those of the normal flow. The resulting reduction in superelevation would differ then in magnitude from that of the normal case.

III. LABORATORY INVESTIGATION

The nature of the apparatus used in the experimental work may best be understood by reference to the photographs in Figs. 7 and 8 and the sketch in Fig. 9. These show the general arrangement of the apparatus at both the University of Iowa and Wayne University. The two sets of equipment were essentially the same in principle, providing for the introduction of a jet of water rectangular in cross-section to a channel with walls which formed a transition of the desired type. It was assumed that only one-half of the channel need be used, the center-line being simulated by a plate of glass forming a straight vertical wall, and the curved wall being simply a piece of pyralin so constructed that it could be formed to the desired plan and held in place.

As may be seen in Figs. 7 and 9, the source of water at the University of Iowa was a ten inch distribution pipe coming from the constant head tanks. The flow from this pipe was controlled by a six inch valve located in a pipe leading from a tee in the main line. The water passed through a weir box equipped with a 90° v-notch weir and hook gage to a channel in which the flow was quieted by means of baffles, thence through a device which provided a means for varying the size of the jet of water introduced to the channel. The jet issued from a tube, rectangular in cross-section with contractions suppressed

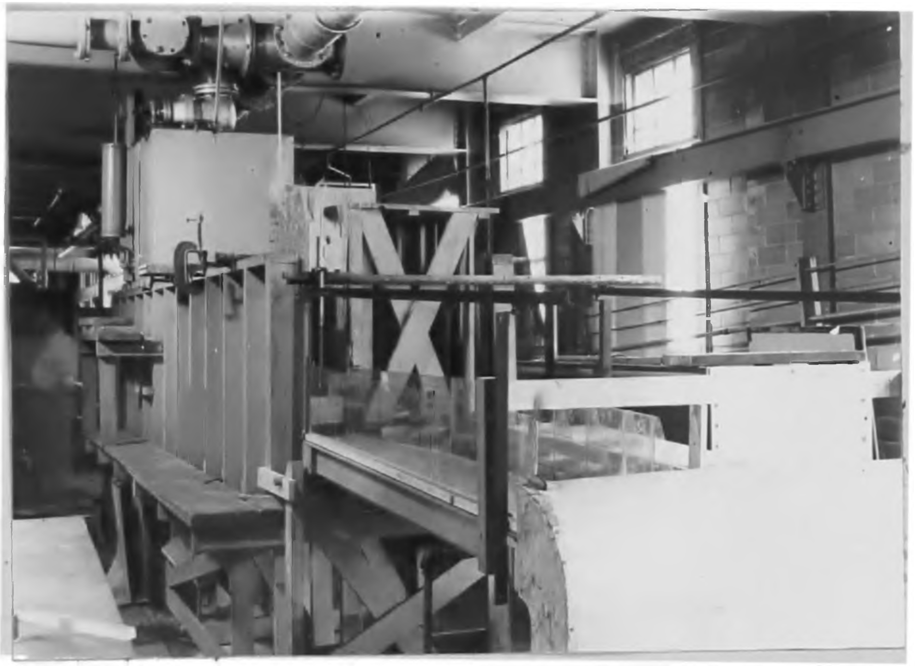


Fig. 7

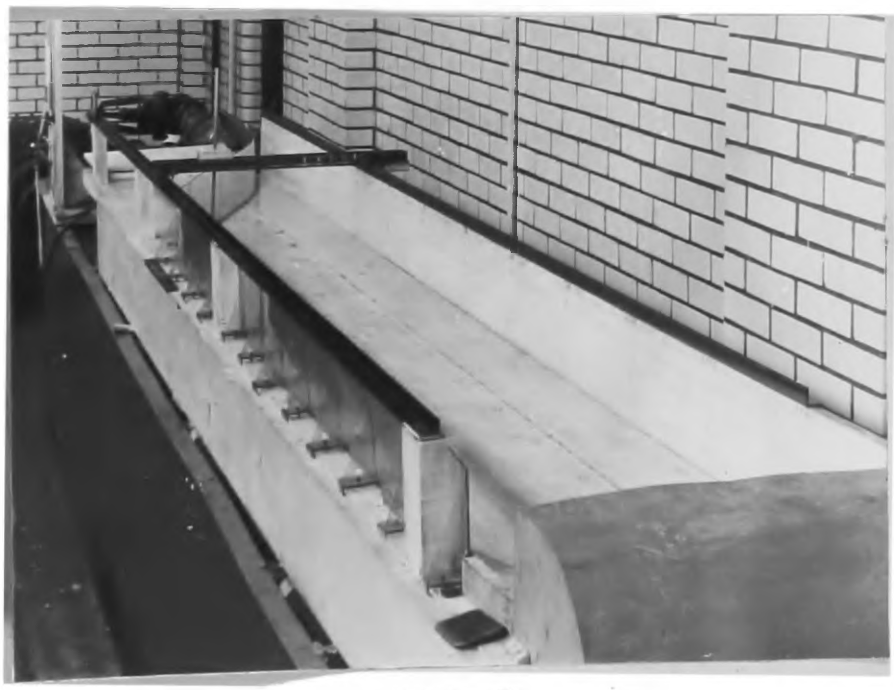
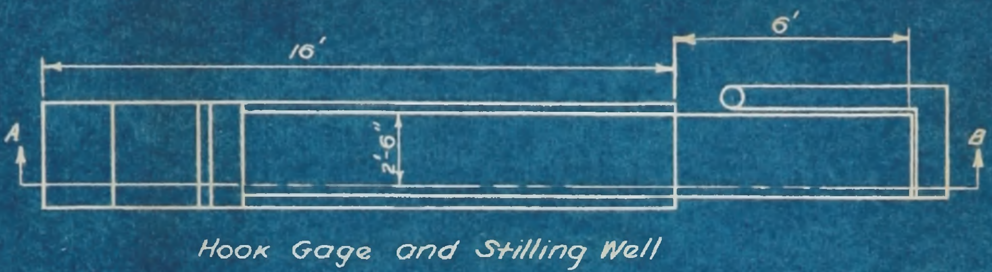
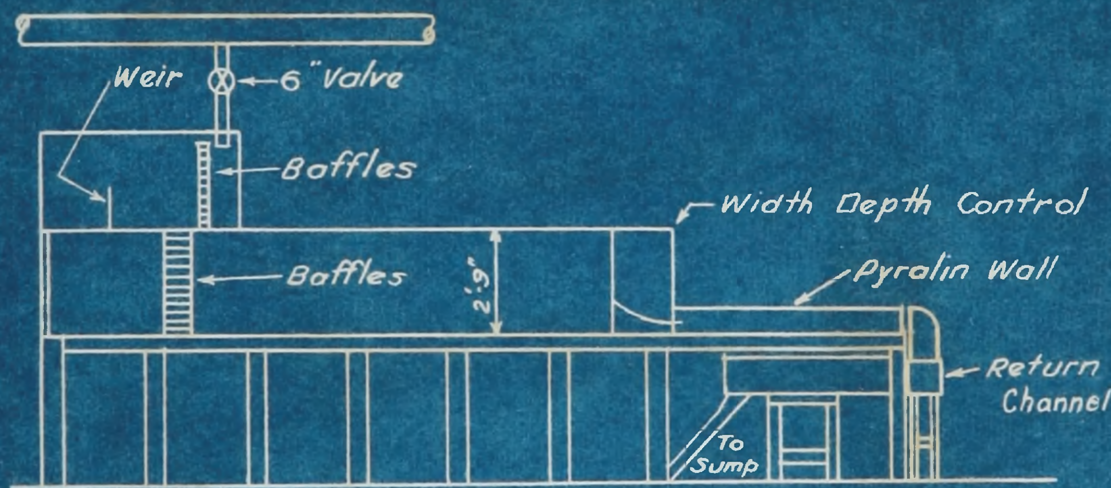


Fig. 8



PLAN



SECTION A-A

PRINCIPAL FEATURES OF
APPARATUS
AT
UNIVERSITY OF IOWA

FIGURE 9

on two sides by means of cylindrical surfaces of one foot radius, the other two sides being the bottom and one side of the approach channel. This equipment made possible the production of a jet with minimum dimensions of 1" x 1" and maximum dimensions of 6" x 6". All combinations of the intermediate depths and widths were possible but limitations on the quantity of water and velocity available fixed the maximum size at about twenty-four square inches in cross-sectional area. The transition section was constructed on a horizontal plane surface which consisted of composition board mounted on a suitable structural frame, painted and marked off in a rectangular coordinate system. Details of a transition section may be seen in Fig. 10.

The apparatus used at Wayne University differed in one essential respect from that described above -- no adjustment of the size of jet was possible. The jet was formed by means of a galvanized iron transition piece from a three inch circular pipe to a one and one-half inch square tube. The jet issued directly upon a horizontal surface formed by a two-inch plank mounted on a structural frame. The surface was painted and marked in a rectangular coordinate system as was the one at the University of Iowa. The discharge was determined by means piezometers in the transition section of the tube which was calibrated in place. This tube is shown in greater detail in Fig. 11. The water was pumped directly to this tube by a centrifugal

Fig. 10

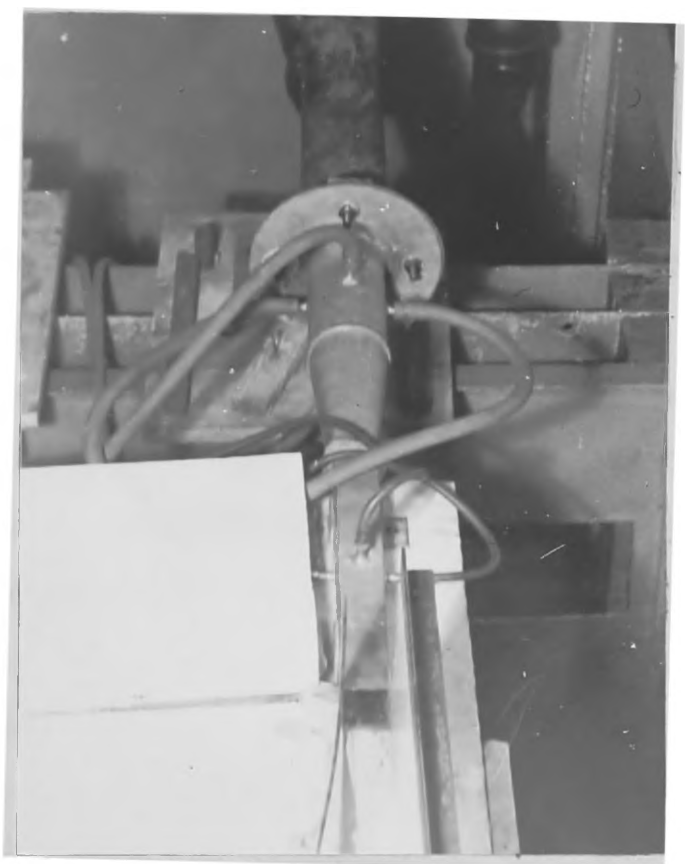


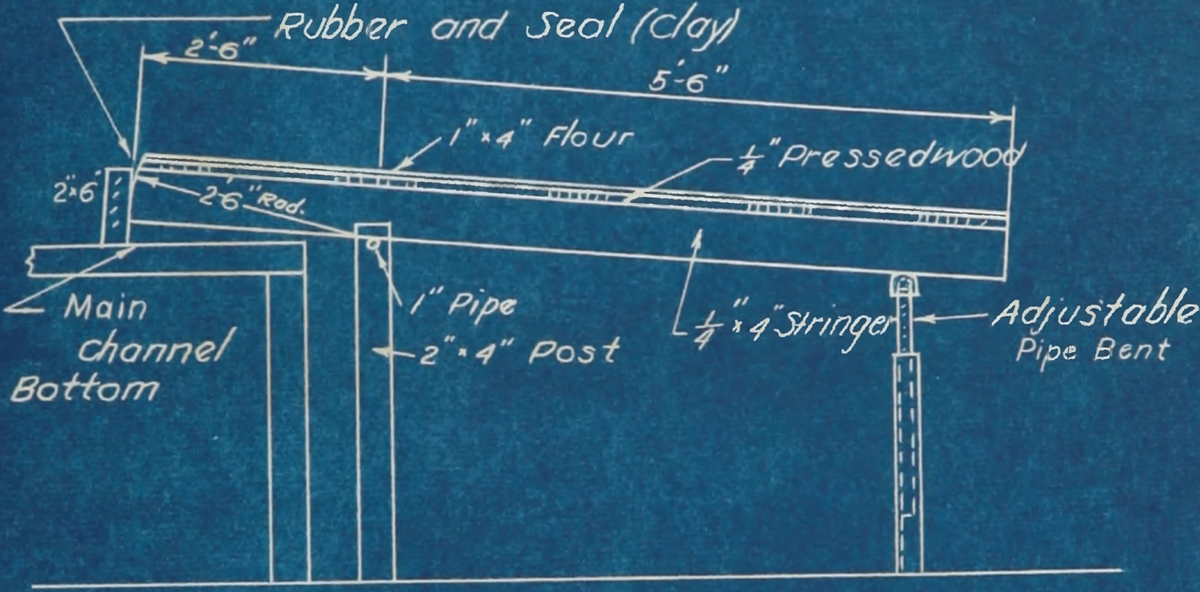
Fig. 11

pump from a sump with no provision for constant head maintenance. No appreciable head variation was noted.

At both the University of Iowa and Wayne University the depth of the water was determined by means of a point gage mounted on a traveling bar which gave full freedom of motion in a horizontal plane. The location of the gage was determined in terms of two coordinates from the origin of the coordinate system marked on the channel bottom by means of graduated tapes and indicators attached to the traveling bar and gage itself. The elevation of the water surface was determined by point gage readings on the surface and channel bottom.

With the apparatus described above it was possible to determine the quantity of water flowing in the channel and the depth at any point within the boundaries of the transition. By direct computation the location of surface contours and the shape of surface profiles could then be determined.

Additional equipment was used in several studies to secure certain information in addition to that which could be obtained with the apparatus described above. This equipment consisted of a channel with sloping bottom illustrated in Fig. 12, piezometers in channel bottom and walls and threads attached to the channel bottom, the latter providing both visual and photographic indications of the direction of the velocity vectors at specified



ADJUSTABLE SLOPE
CHANNEL BOTTOM

FIGURE 12

points in the channel.

In a majority of the runs, notably those made at the University of Iowa during the summer of 1939, the object was simply to determine the elevation of the liquid surface within and beyond the transition section. The procedure under these circumstances was simply to establish the conditions desired by forming the wall to the requisite form, providing a jet with the necessary cross section and establishing a flow with the specified Froude number, and then to make as many observations with the point gage on the water surface as was deemed advisable.

In later runs, when the importance of the pressure distribution had been realized, more numerous observations on the elevation of the surface of the liquid were made and the pressures at certain specified points were determined by means of piezometers.

IV. ANALYSIS OF EXPERIMENTAL DATA

(a) Transitions with reversal of wall curvature

The essential findings of this study are summarized briefly in Fig. 13 wherein is shown the ratio of the maximum depth along the curved wall d_m to the initial depth expressed by the ratio d_m/d_o as a function of the initial Froude number, with the pertinent geometrical ratios indicated in the legend. In Fig. 4 is shown the nomenclature used herein on a sketch of typical transitions. The use of the ratios m/b_o and b_e/b_o was based on a study of the influence of the fundamental ratios m/d_o, b_o/d_o and b_e/d_o which indicated that the use of the product $m/d_o \times d_o/b_o = m/b_o$ was justified since all points representing data based on transitions with equal values of this product of length ratios lay on the same straight line. A similar situation existed with respect to the ratio b_e/b_o .

The principle feature of this diagram is the fact that it conforms to expectations on the basis of the elementary wave theory in a general manner. The ratio d_m/d_o exceeds the value unity for only a few cases which may be explained on the basis of high local vertical accelerations at the downstream point of tangency between curve and straight wall. At low Froude numbers the value of d_m/d_o is low, since the negative waves, making large angles with the original direction of flow, cross the channel and reduce the superelevation of the water surface. In long curves, which are indicated by large values of m/b_o, a


1.0
0.5
0
$\frac{d_m}{d_o}$
0
10
20
30
$\frac{V_o^2}{g d_o}$
$\frac{d_o}{b_o}$ $\frac{m}{b_o}$ $\frac{b_e}{b_o}$
1 1/4 2 2
2 1/4 4 2
3 1/4 4 3
4 1/4 4 4
5 1/2 4 4
6 1/2 8 4
7 1/2 8 3
8 1/2 4 3
9 1/2 4 2
10 1/2 2 2
11 1/2 2 3
12 1 8 3
13 1 8 4
14 1 4 3
15 1 4 2
16 1 4 4
17 1 5 5
18 1 10 4
19 2 8 4
20 2 8 3
21 2 20 16
22 2 8 2
23 2 4 2
24 2 4 3
25 3 10 4
26 3 10 8
27 3 20 8
28 3 20 16

FIGURE 13

similar effect is noticed, since the wave fronts, even though making small angles with the wall, cross the channel within the limits of the transition and reduce the super-elevation.

The flow in the upstream portion of this type of transition is identical with that in a transition without reversal of wall curvature. The analysis of the data for this part of the channel will be carried out in the next section which relates only to flow in a channel bounded by diverging curved walls and straight tangents.

A quantitative analysis of the reduction in super-elevation by the waves crossing the channel is not feasible, since the flow with subnormal pressure is not yet subject to complete analysis even when the waves do not cross within the limits of the transition.

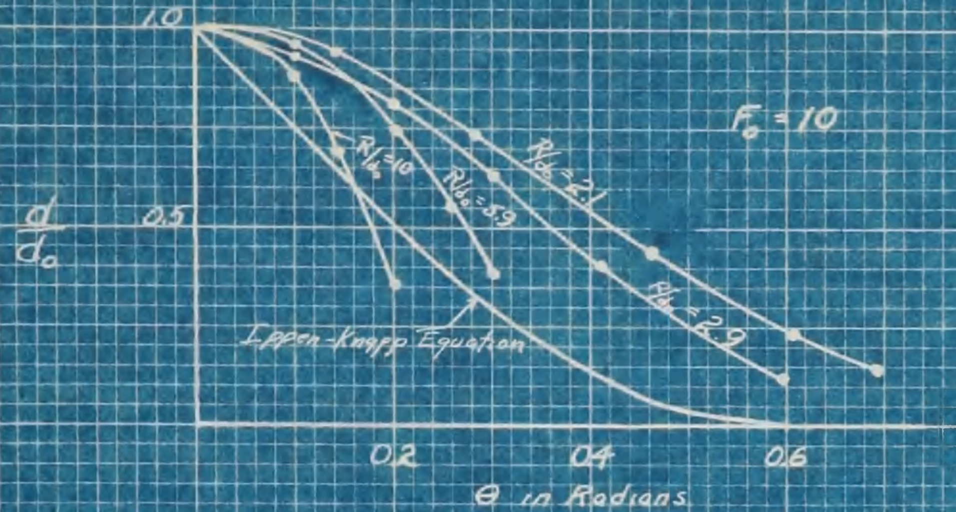
(b) Flow at diverging curved wall

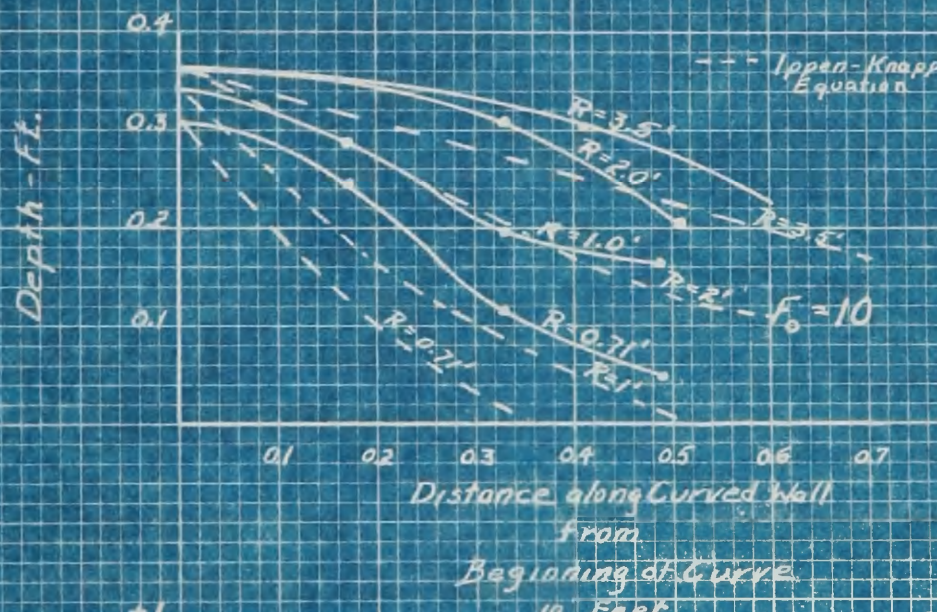
The theory of the propagation of small disturbances in a liquid flowing at supercritical velocity was extended to include the situation wherein subnormal pressures exist within the body of the liquid in an effort to explain differences which became evident during the investigation between actual surface profiles and those predicted by the theory based on normal pressure distribution. In order to determine the explanation for these differences between the predictions of the elementary theory and actual observation it is necessary to establish certain facts.

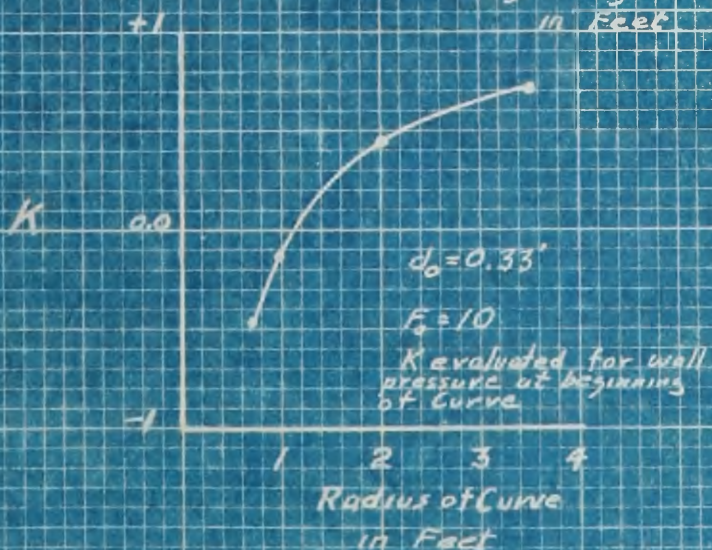
Since the elementary theory assumed that the friction effects were negligible and the pressure distribution was hydrostatic, lack of conformity with either of these assumed conditions would merit consideration. It has been shown previously that the friction at the boundaries would tend to increase the surface elevation. The existence of a non-hydrostatic pressure distribution could have the effect of either raising or lowering the elevation above the normal, depending upon the relative radius and the Froude number.

It remains therefore to either establish the frictional drag as the explanation for the difference between normal and actual profiles or to eliminate it from consideration and to establish or disprove the existence of subnormal pressures and discuss the possible effects if the existence of such pressures is demonstrated. In order to carry out the necessary analysis of the experimental data the plottings shown in Figs. 14, 15, 16, and 17 were prepared.

In studying the effect of frictional drag it is first necessary to establish, if possible, the probable magnitude of the surface slopes due to this factor alone. Secondly, the relative magnitude of the effect of friction in various cases should be evaluated. The first of these points is simply handled by computing the magnitude of the surface slope for a general case by means of the varied



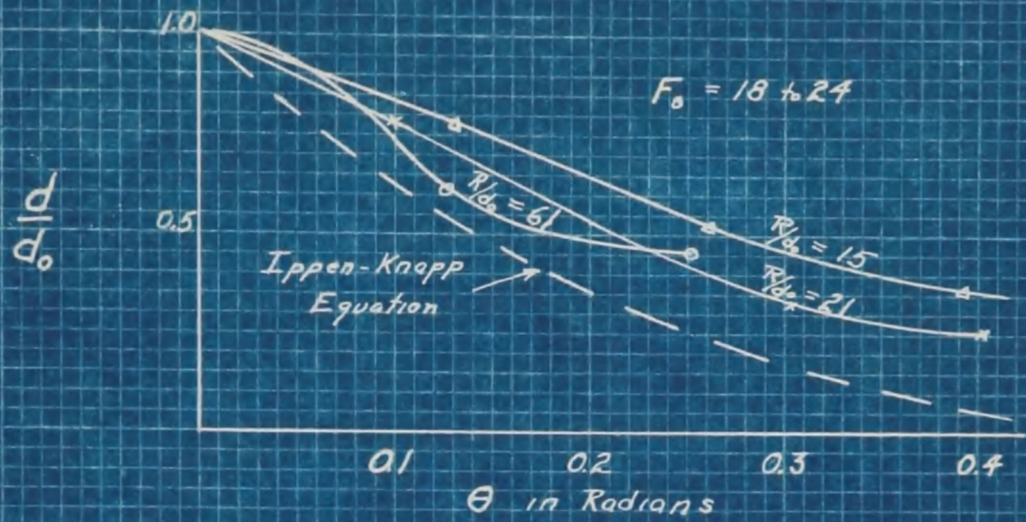




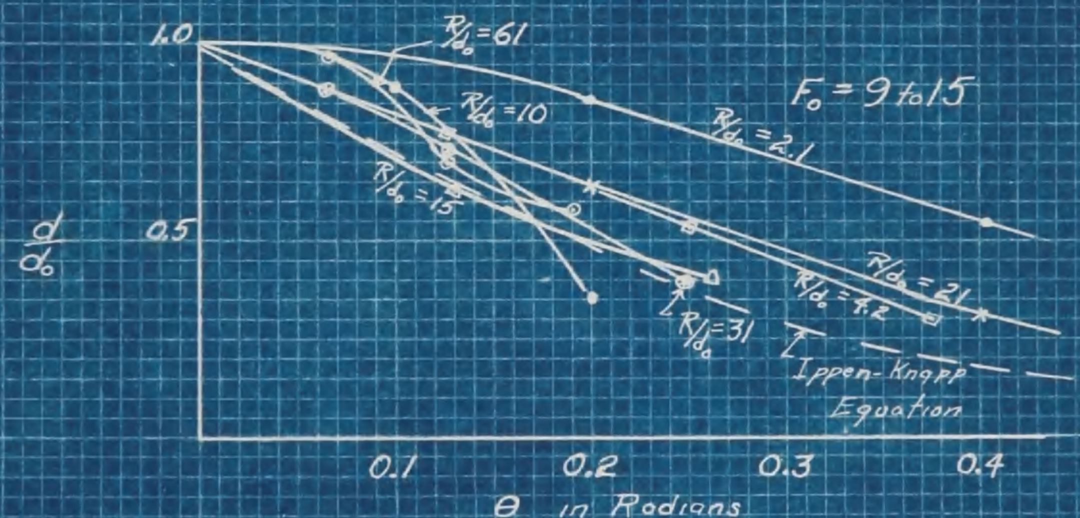
SURFACE PROFILES AT CURVED WALL

RUNS 175-178 AND K as a Function of R

FIGURE 14

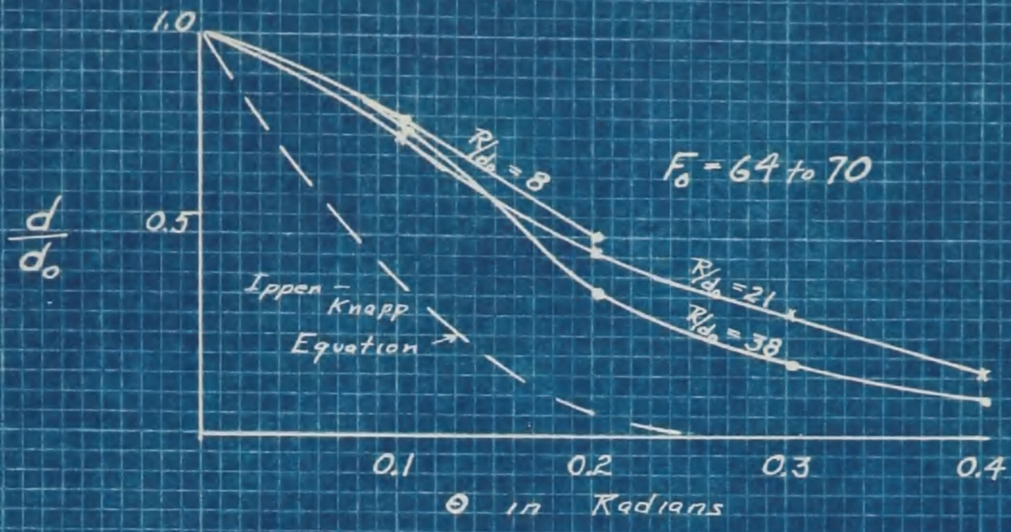


Note: θ is the total angle between the wall and the original straight tangent.

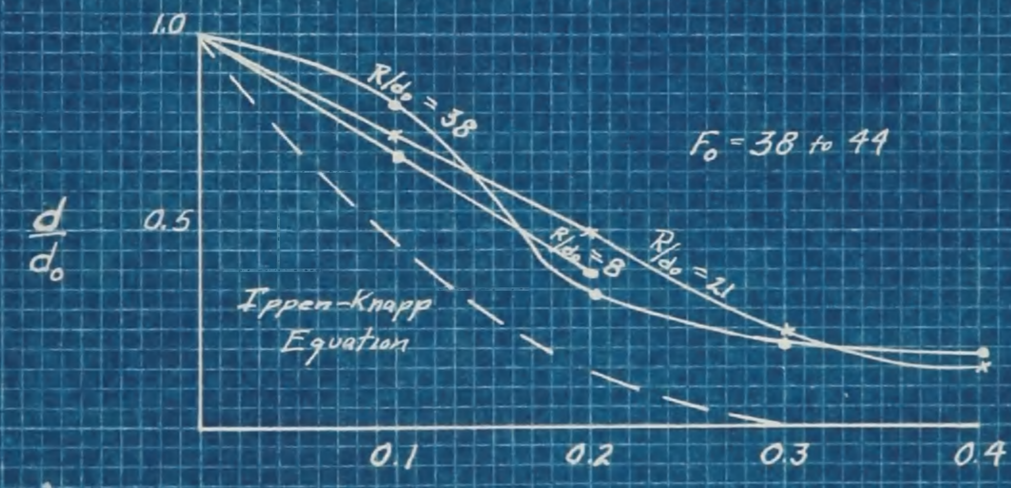


SURFACE PROFILES
AT THE
CURVED WALL
IN
DIMENSIONLESS FORM

FIGURE 15

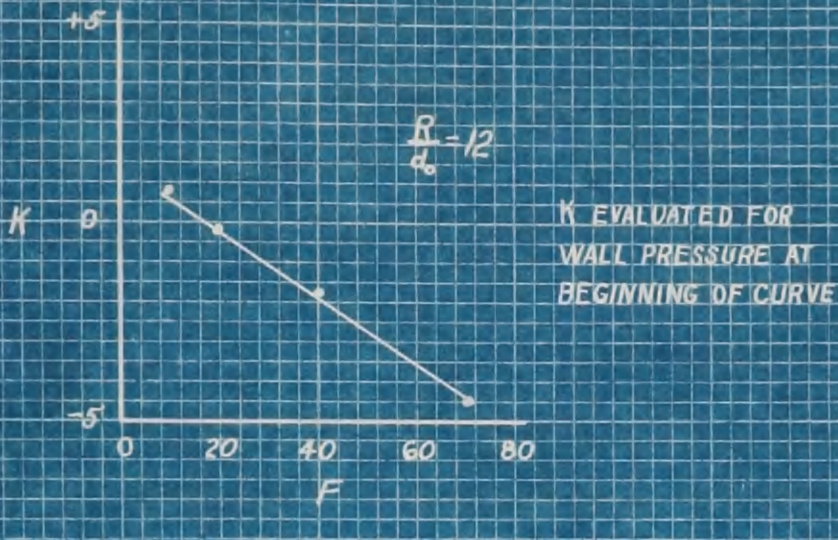


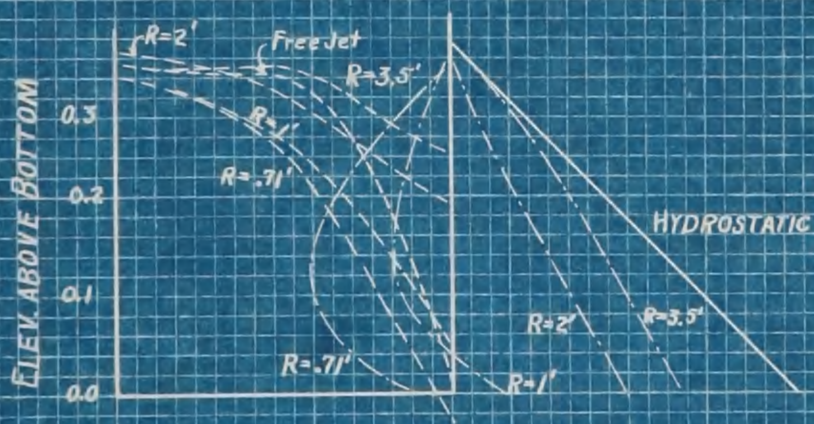
Note: θ is the total angle between the wall and the initial straight tangent.



SURFACE PROFILES AT THE CURVED WALL IN DIMENSIONLESS FORM

FIGURE 16





WALL AND BOTTOM PRESSURES FOR SECTION
AT BEGINNING OF CURVE

FIGURE 17

flow equation. As has been shown previously, this equation reduces to the following form for large Froude numbers with flow taking place on a horizontal surface,

$$\frac{dd}{dx} = \frac{-u^2/\ c^2R}{1 - u^2/\ gd}$$

This equation may also be written more simply in the following form,

$$\frac{dd}{dx} = \frac{-u^2/\ c^2R}{1 - F}$$

This equation may be used to determine approximately the surface slope due to frictional drag alone. The approximation lies chiefly in the value of Chezy's coefficient, tabulated values of which have been evaluated for uniform flow. The second point is concerned chiefly with the fact that the longer the curved surface along which the liquid is in contact the greater should be the total effect of the frictional drag.

To study these two points Figs. 14, 15, and 16 will be useful. In Runs 175-178 the initial Froude number was 10, the initial depth was approximately 0.33 ft., and the initial velocity 10 ft. per sec. The value of Chezy's C was not determined but for rough calculations the value 100 will be found to be a fairly good average figure. The hydraulic radius would be initially approximately 0.1 ft. and would not decrease materially for a considerable distance downstream. The surface slope due to friction would be then approximately

$$\frac{dd}{dx} = 0.01$$

We should expect then in the distance 0.25 ft. from the beginning of curve to a point downstream that there would be accumulated due to friction alone an increment in elevation above the normal of approximately 0.003 ft. Actually we observe the following,

R ft.	Actual d ft.	Normal d ft.	Increment ft.
0.71	0.18	0.064	0.116
1.0	0.24	0.128	0.112
2.0	0.33	0.250	0.080
3.5	0.34	0.290	0.050

These increments are not only much larger than those indicated for friction alone but their variation with relative radius is in the direction opposite to that which would obtain were these increments due to friction alone. The latter feature becomes even more marked if we consider greater distances downstream. Eventually the longest radius curve shows a surface profile dipping below the normal profile. Additional evidence on these two points may be obtained from the data in Figs. 15 and 16. In the case of the very large Froude numbers one might expect a more pronounced friction effect. Consider for example a Froude number of 70, a depth of 0.13 ft., an hydraulic radius of

0.04 ft. and a velocity of 17 ft. per sec. The slope due to friction in this case would be again 0.01. It is obvious without further calculation that the observed differences between normal and actual profiles exceed those which would be caused by friction. No definite trends with respect to the relation between the relative radius and the superelevation in the case of the high Froude numbers is observable. The other plottings for lower Froude numbers shown in Figs. 15 and 16 present similar evidence. It is clear from the available data that a frictional drag is insufficient to explain the observed differences between normal and actual depths.

The existence of subnormal pressures in the flows investigated is established by the data shown in the plottings of the individual Runs 175-178 as well as by the summary of these data in Fig. 17. In Runs 175-178 values of k ranged from +0.7 to -0.47. In previous observations of lesser accuracy values of k as low as -4.0 were observed. In Runs 175-178 the observations for k were made at a vertical section as close as was possible to the beginning of the curve.

Subnormal pressures of such a magnitude that they were actually sub-atmospheric were first observed in experiments performed with the apparatus at Wayne University. There was at first some doubt concerning the accuracy of the piezometer readings which indicated the sub-atmospheric pressures. The possibility of irregularities in the apparatus was considered and thoroughly investigated and the observations repeated. The results were substantially the same. Several months later the runs numbered 174-178 were made at the University of Iowa with the results shown in the plottings of Figs. 18, 19, 20, 21, and 22. The sub-atmospheric pressures were again observed although they were of a lesser magnitude. The lesser magnitude was to be expected since the Froude number was only 10 in that series of runs whereas it had been as high as 70 in the runs made at Wayne University.

An explanation of the existence of a sub-atmospheric pressure close to the free surface of a body of liquid has not been undertaken at this time. It may well be explained by the fact that the pressure asymptotically approaches atmospheric as the distance from the free surface decreases. However, this has not yet been fully established and serves only as a possible working hypothesis.

The effect of subnormal pressures may best be understood by reference to Fig. 3. Assume for example that for $F_o = 100$ and a certain relative radius the reduction of k from its initial value of unity to a value of 0.8 occurs with $\Delta k/\Delta\theta = 2$, and for a shorter relative radius and identical Froude number the reduction is to $k = 0.6$ with $\Delta k/\Delta\theta = 4.0$. It is apparent

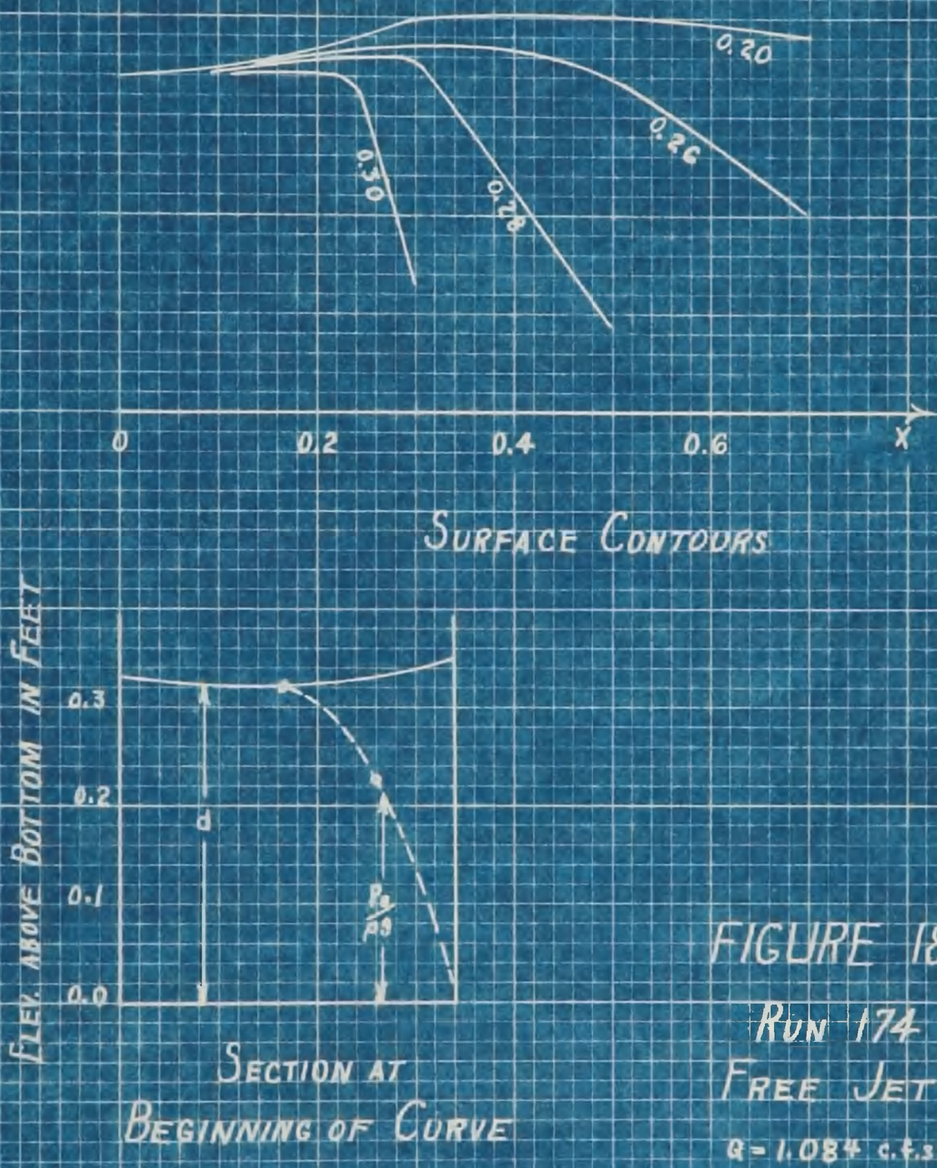


FIGURE 18

RUN 174

FREE JET

$Q = 1.084$ c.f.s.

$A_0 = 0.112$ sq. ft.

$V_0 = 9.7$ f.p.s.

$F_0 = 10$

$d_0 = 0.33$ ft.

$b_0 = 0.34$ ft.

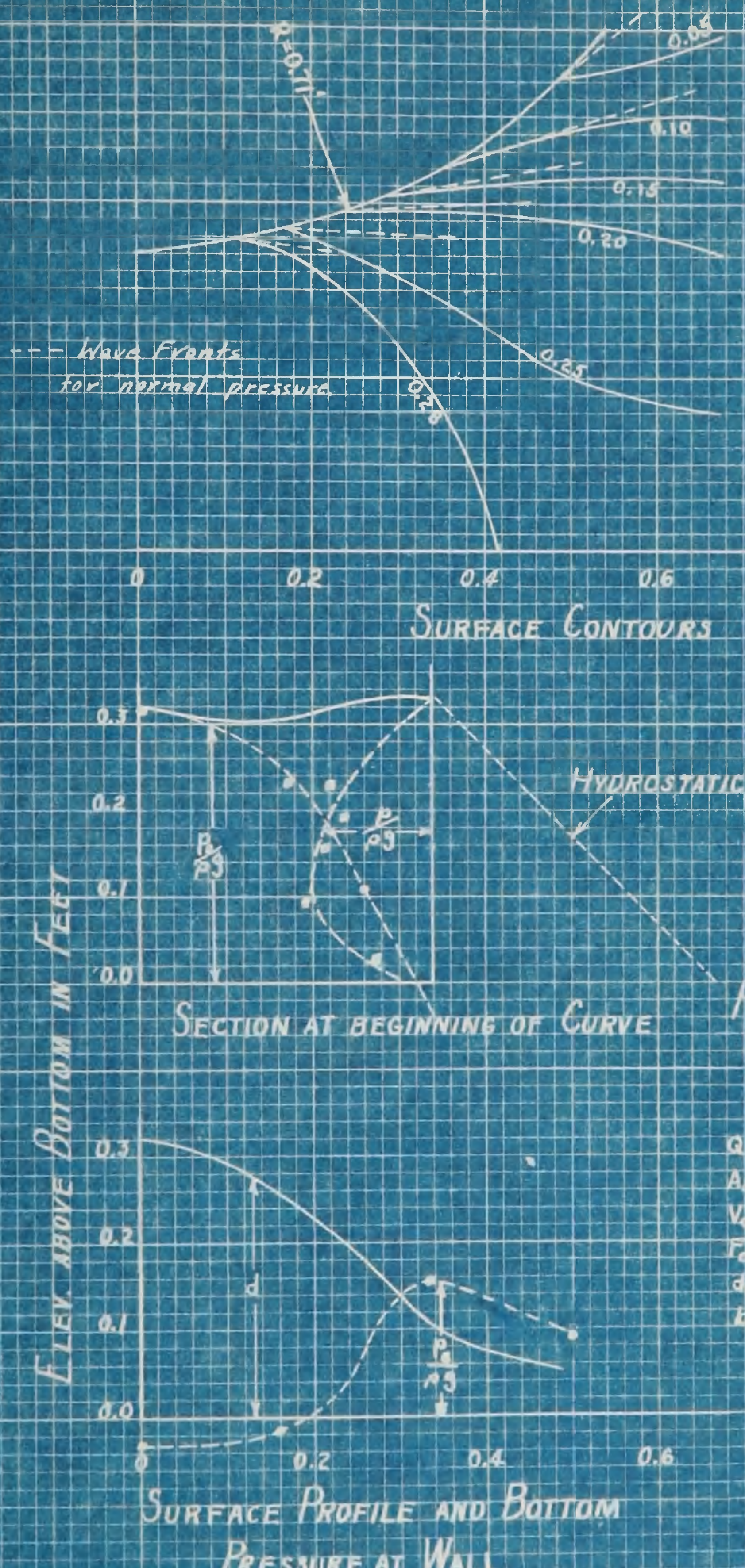


FIGURE 19

Run 175

$Q = 1.084$ c.f.s.
$A_o = 0.106$ sq. ft.
$V_o = 10.2$ f.p.s.
$F_o = 10$
$d_o = 0.31$ ft.
$b_o = 0.34$ ft.

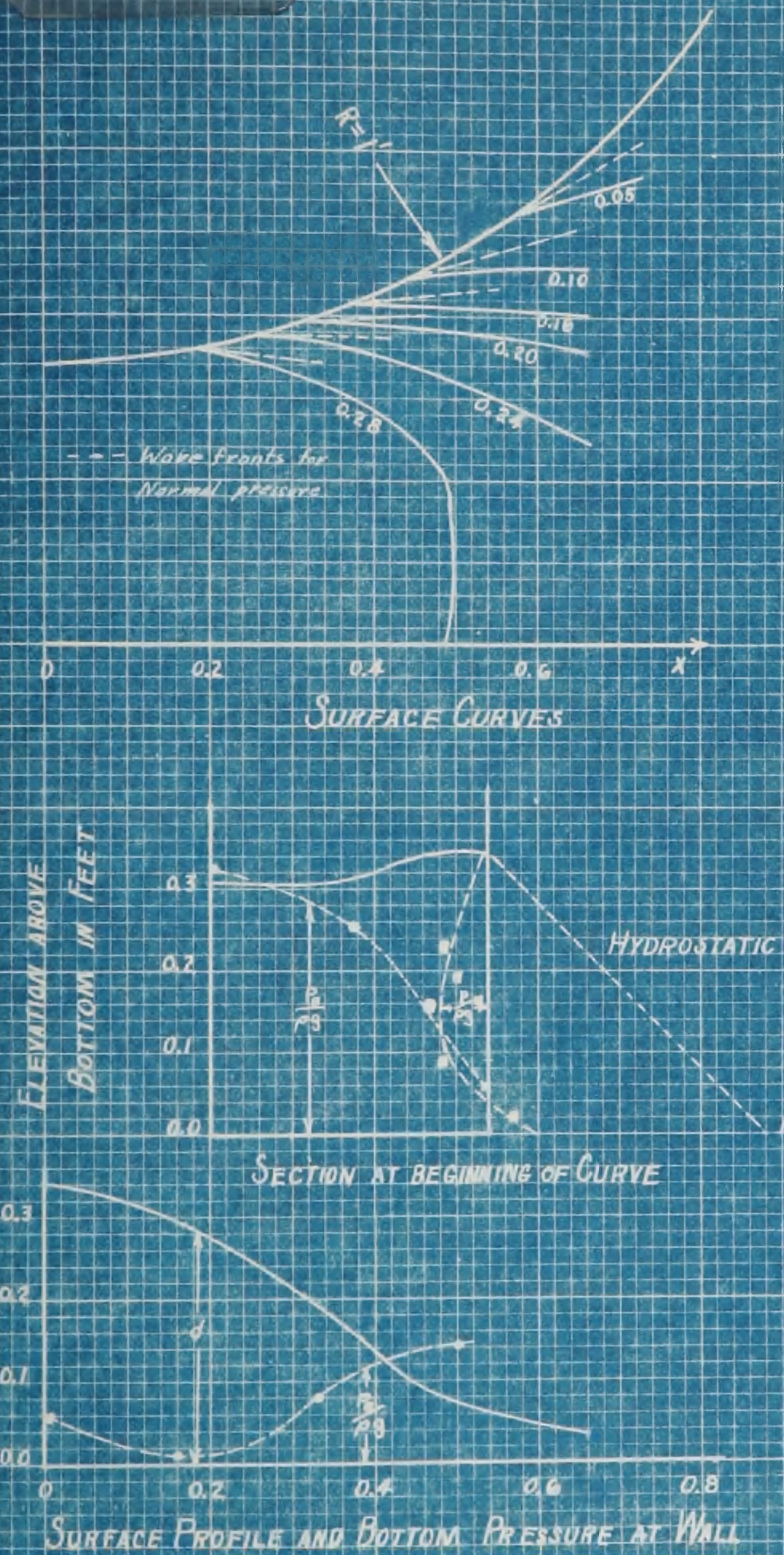


FIGURE 20

RUN 176

$Q = 1.084$ cfs.

$A_o = 0.109$ sq. ft.

$V_o = 10.0$ f.p.s.

$F_o = 10$

$d_o = 0.32$ ft.

$b_o = 0.34$ ft.

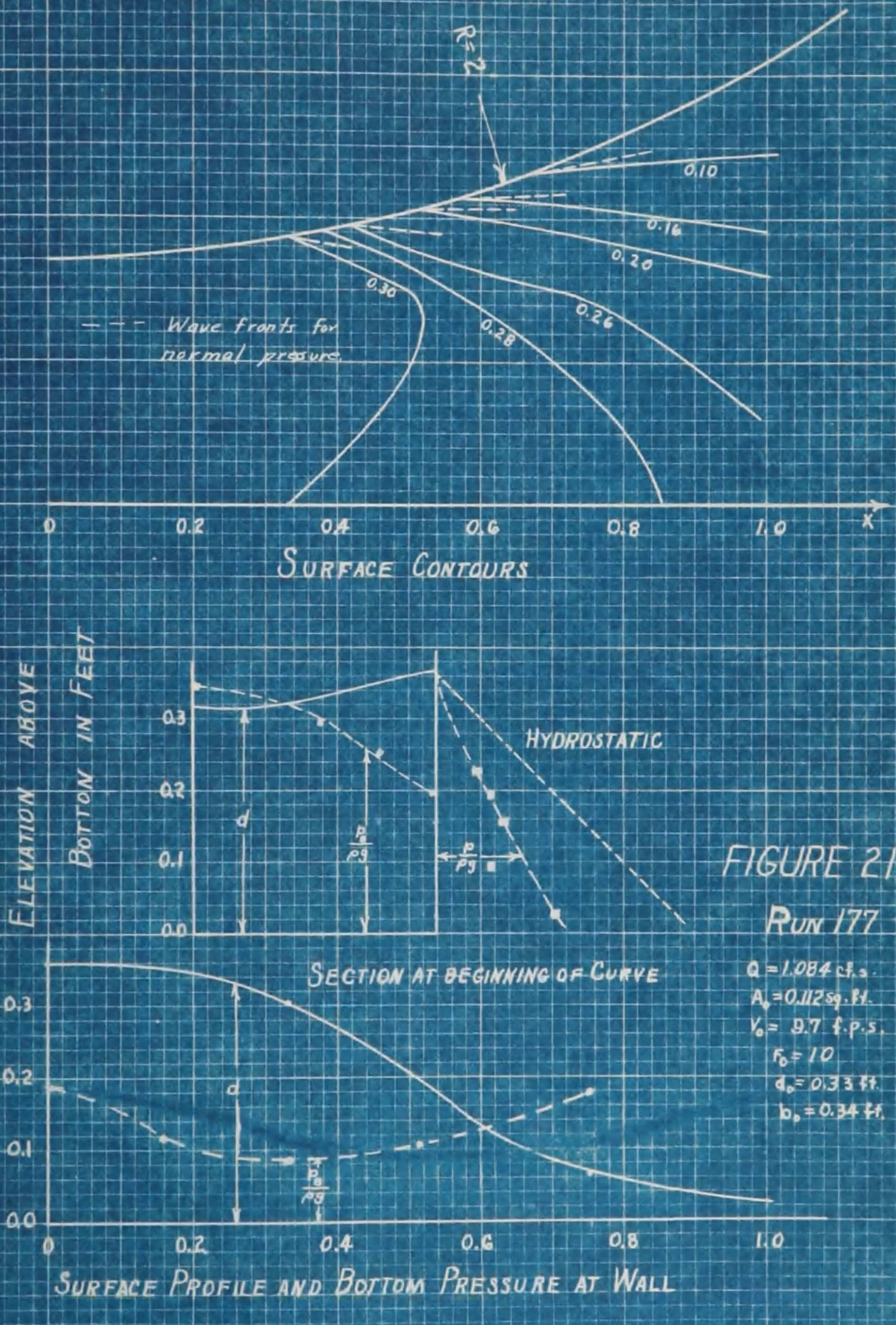


FIGURE 21

RUN 177

$Q = 1.084$ c.f.s.
$A_0 = 0.112$ sq. ft.
$V_0 = 9.7$ f.p.s.
$F_0 = 10$
$d_0 = 0.33$ ft.
$b_0 = 0.34$ ft.

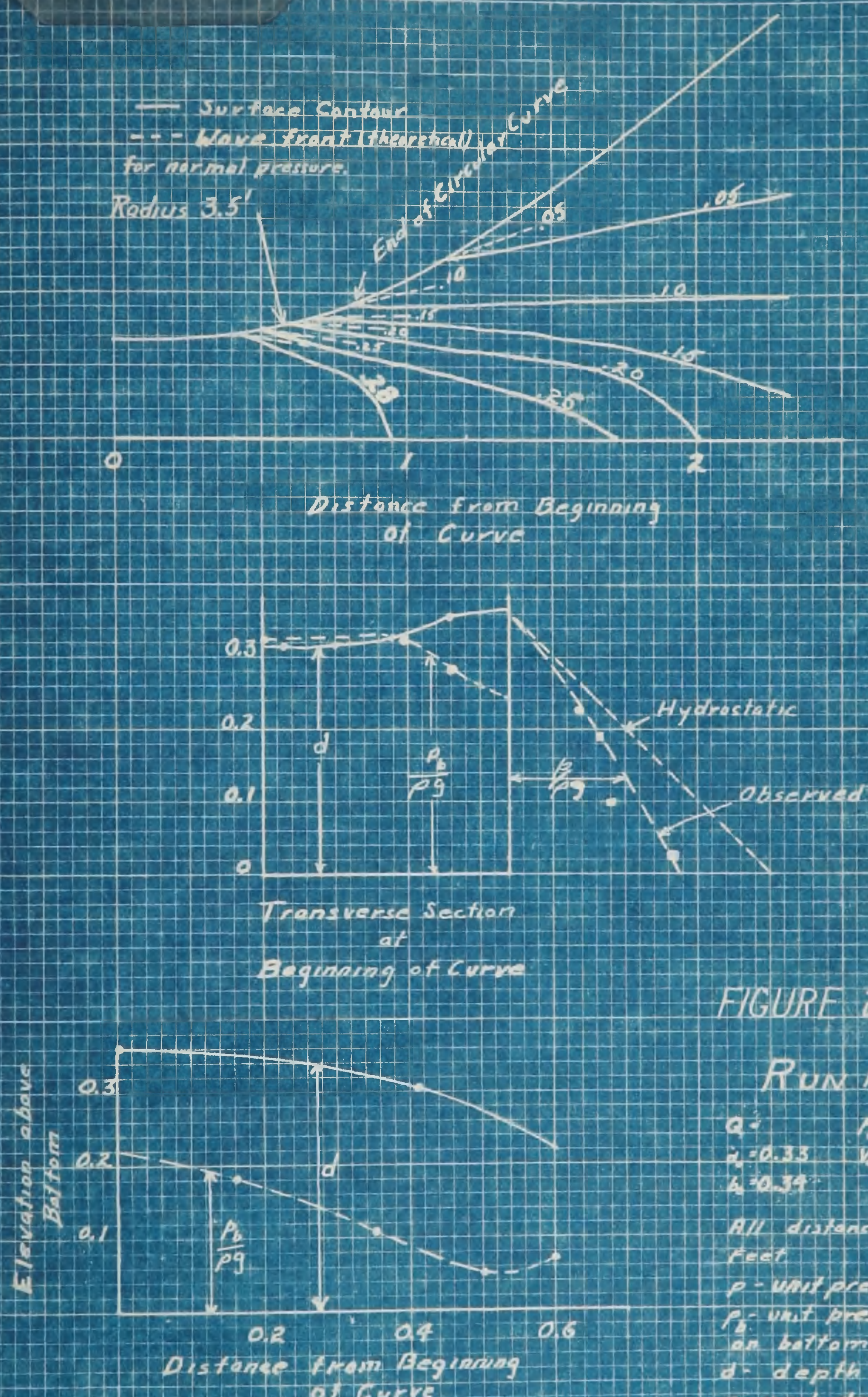


FIGURE 22

RUN 178

Q = F_0 = 10.
d_0 = 0.33 V_0 = 10
b_0 = 0.34

All distances in feet
p - unit pressure
P_b - unit pressure on bottom
d - depth
ρ - unit density

from the diagram that the resulting depression of the surface will be greater for the sharper curvature, for a given $\Delta\theta$ assuming the change in F to be small. However a similar set of values for k and $\Delta k/\Delta\theta$ would yield an entirely different result for an initial Froude number of 10. In this case the situation would be reversed and the lesser curvature would give the greater depression. The latter situation is apparent in Runs 175-178. It follows therefore that the extended theory provides a qualitative basis for the explanation for a portion of the observed effects of subnormal pressure.

The variation of k with the relative radius and with F is indicated from the available data as shown in Figs. 14 and 17. Short relative radii give small values of k and high Froude numbers give small k's . The value of k approaches unity as the relative radius increases and as the Froude number approaches unity.

A comparison of the surface contours of Runs 174 and 175 indicates that the reduction of elevation of the surface is effected more rapidly with a short radius wall in place than in the free jet.

V. SUMMARY AND CONCLUSIONS

(a) Summary

The original objective of the investigation which was the establishing of principles and the formulation of a procedure to be used in the design of transitions in rectangular open channels in which the flow would be at supercritical velocities, was modified to the extent described below. The general characteristics of the flow in a transition with reversal of wall curvature were investigated thoroughly and on the basis of that study it was concluded that although the general behavior was in accord with the theory a quantitative prediction of the elements of the flow was not feasible at this time for two reasons. First of these was that the flow was actually very complex and could not always be broken down into simple flows the elements of which could be predicted. The second reason was that the assumption of hydrostatic pressure distribution used in the elementary theory and equations developed by Ippen and Knapp was not fulfilled. As a consequence of the latter fact neither the elements of the flow in the upstream portion of a transition with reversal of wall curvature nor those of the ensuing downstream flow, which was directly affected thereby, were subject to quantitative prediction.

An analysis, both theoretical and experimental in character, was made of the flow along a diverging curved wall assuming non-hydrostatic pressure distribution to determine the characteristics of the surface profile. The resulting equations provide an explanation of the differences between the observed surface profiles and those predicted by the theory based upon the assumption of hydrostatic pressure distribution.

The extension of the theory of the propagation of small waves in a liquid flowing at supercritical velocity to include the situation involving non-hydrostatic pressure distribution and the demonstration that pressures less than hydrostatic and indeed sub-atmospheric may exist in a flow bounded by diverging curved walls are the principle features of the investigation.

(b) Conclusions

The summary of the findings related to transitions with reversal of wall curvature and the conclusion that in general the flow with that type of boundary is too complex for complete analysis at present are set forth above.

The following conclusions summarize the findings related to the flow bounded by simple diverging curved walls. The word normal is used to refer to the elements of the flow predicted by the elementary theory based upon the assumption of hydrostatic pressure distribution and negligible frictional effects.

1. Experimental observations establish the fact

that there is a difference between the actual profiles of the liquid surface at the curved boundary and those predicted by the theory based upon the assumption of negligible frictional effects and hydrostatic pressure distribution.

2. The experimental data indicate that the differences between the normal and actual surface elevation are of a magnitude which cannot be explained by the frictional drag.

3. The effect of a relatively long radius in producing a large relative increase in the elevation due to frictional drag was not observed. Occasionally long relative radii were accompanied by less superelevation above normal elevation than was found for short relative radii.

4. Subnormal pressures were observed at the wall and in the body of a liquid flowing adjacent to a curved wall.

5. The extended theory of the propagation of small disturbances in a liquid flowing at supercritical velocities indicates the same type of differences between actual and normal profiles as were observed.

6. The presence of a sharply curved wall may effect a more rapid reduction of the surface elevation of a liquid flowing on a horizontal surface than could be accomplished in a free jet flowing on the same surface at the same initial Froude number.

7. The existence of subnormal pressures, which

may be actually sub-atmospheric, at the curved wall of an open channel transition creates a structural problem which should be considered in the design of such structures.

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APPENDIX

(a) Derivation of Equations

1. Wave velocity with hydrostatic pressure distribution

For the sake of simplicity the artifice of superimposing upon the liquid a uniform motion having the velocity of propagation of the small wave will be used. This transforms a problem of unsteady flow to one of steady flow. We shall consider a negative wave, that is one arising from a depression of the water surface. This wave would move to the left in Fig. 2. The entire body of fluid will then be imagined to be moved to the right at the velocity c thus holding the wave form motionless. The liquid will then appear to flow through the wave form towards the right.

It is assumed that the velocity is uniform throughout the depth, the pressure distribution is hydrostatic, and the frictional losses are negligible.

The momentum equation is written for sections 1 and 2 of Fig. 1 in this manner:

$$P_1 - P_2 = \rho\,(c_2^2 d_2 - c_1^2 d_1)$$

The continuity equation gives the following information,

$$c_1 d_1 = c_2 d_2$$

Since the pressure distribution is hydrostatic it follows that

$$P_1 - P_2 = \frac{\rho g}{2}\,(d_1^2 - d_2^2)$$

Substitution of this value for $(P_1 - P_2)$ above gives

$$\frac{g}{2}[d_1^2 - d_2^2] = c_2^2 d_2 - c_1^2 d_1$$

and upon elimination of c_2 by means of the continuity equation there results,

$$\frac{g}{2}[d_1^2 - d_2^2] = \frac{c_1^2 d_1^2}{d_2} - c_1^2 d_1$$

which may be written

$$\frac{g}{2}[d_1 + d_2][d_1 - d_2] = \frac{c_1^2 d_1}{d_2}[d_1 - d_2]$$

Dividing both sides of this equation by $d_1 - d_2$ we have:

$$\frac{g}{2}[d_1 + d_2] = \frac{c_1^2 d_1}{d_2}$$

which may be simplified by taking note at the assumption of a small wave in which case d_1/d_2 is approximately unity and $(d_1 + d_2)$ is approximately 2d. The simplified form is,

$$gd = c^2 \qquad (1)$$

or in a more familiar form

$$c = \sqrt{gd}$$

2. Depth change upon passing under a wave front

From the momentum equation given above in the derivation of Equation (1) we have upon substitution of Δd for $d_1 - d_2$,

$$\frac{g}{2}\left[d^2-(d-\Delta d)^2\right]=(c+\Delta c)^2(d-\Delta d)-c^2d^2$$

which upon expansion and elimination of terms of higher order becomes:

$$\frac{g}{2}\left[d^2-d^2+2d\Delta d\right]=-c^2d+c^2d+2c\Delta cd-c^2\Delta d$$

and simplifies to

$$g\,d\,\Delta d = 2c\,\Delta c\,d - c^2\Delta d$$

or

$$\Delta d\,(g\,d+c^2) = 2\,c\,\Delta c\,d$$

We have shown that $c = \sqrt{gd}$, therefore

$$2\,g\,d\,\Delta d = 2c\,\Delta c\,d$$

or

$$g\,\Delta d = c\,\Delta c$$

Now from the geometry at the vector diagram

$$c = u\sin\beta$$

and

$$(c+\Delta c)/\tan(\beta+\Delta\theta) = u\cos\beta$$

or

$$\Delta c = \frac{u\,\Delta\theta}{\cos\beta}$$

Substituting for c and Δc we obtain:

$$\Delta d = \frac{u^2\sin\beta\,\Delta\theta}{g\,\cos\beta}$$

or

$$\Delta d = \frac{u^2\tan\beta\,\Delta\theta}{g} \tag{4}$$

3. Wave velocity with non-hydrostatic pressure distribution.

The simultaneous solution of Equations (6), (7), and (8) proceeds as follows

$$P_1 = k_1 \frac{\rho g d_1^2}{2} \qquad P_2 = k_2 \frac{\rho g d_2^2}{2} \tag{6}$$

$$P_1 - P_2 = \rho\,(c_2^2 d_2 - c_1^2 d_1) \tag{7}$$

$$c_1 d_1 = c_2 d_2 \tag{8}$$

Upon substituting in (7) the values of P_1 and P_2 from Equation (6) we obtain:

$$\frac{\rho g}{2}\left[k_1 d_1^2 - k_2 d_2^2\right] = \rho\left[c_2^2 d_2 - c_1^2 d_1\right]$$

But from the continuity Equation (8) we have

$$c_2 = \frac{c_1 d_1}{d_2}$$

which permits the following simplification:

$$\frac{g}{2}\left[k_1 d_1^2 - k_2 d_2^2\right] = \left[\frac{c_1^2 d_1^2}{d_2} - c_1^2 d_1\right]$$

This may be simplified considerably in the following manner. Factoring, dividing and multiplying by k, we obtain

$$\frac{g k_1}{2}\left[d_1^2 - \frac{k_2}{k_1} d_2^2\right] = \frac{c_1^2 d_1}{d_2}\left[d_1 - d_2\right]$$

or

$$\frac{g k_1}{2}\left[d_1^2 - d_2^2 + d_2^2\left(1 - \frac{k_2}{k_1}\right)\right] = \frac{c_1^2 d_1}{d_2}\left[d_1 - d_2\right]$$

and upon dividing by $d_1 - d_2$,

$$\frac{gk_1}{2}\left[d_1 + d_2 + \frac{d_2^2}{d_1 - d_2}\left(1 - \frac{k_2}{k_1}\right)\right] = \frac{c_1^2 d_1}{d_2}$$

Assuming $d_1 - d_2$ very small,

$$\frac{gk}{2}\left[2d + \frac{d_2^2}{d_1 - d_2}\left(1 - \frac{k_2}{k_1}\right)\right] = c_1^2$$

which gives as the value for c

$$c = \sqrt{gd\left[k_1 + \frac{k_1 d}{2(d_1 - d_2)}\left(1 - \frac{k_2}{k_1}\right)\right]} \qquad (9)$$

4. Wave angle

Since $\sin\beta = \frac{c}{u}$

we have,

$$\sin\beta = \sqrt{\frac{gd}{u^2}\left[k_1 + \frac{k_1 d}{2(d_1 - d_2)}\left(1 - \frac{k_2}{k_1}\right)\right]} \qquad (10)$$

5. Depth change on passing under a wave front

From momentum considerations we have:

$$\frac{\rho g}{2}\left[k_1 d^2 - k_2 (d - \Delta d)^2\right] = \rho c d \Delta c$$

or

$$\frac{g}{2}\left[k_1 d^2 - k_2 d^2 + 2k_2 d \Delta d\right] = c d \Delta c$$

which becomes

$$\frac{gd^2}{2}\left[k_1 - k_2 + \frac{2k_2 \Delta d}{d}\right] = c d \Delta c$$

Upon rearranging terms we have,

$$\frac{\Delta d}{d} = \left[\frac{2cd\Delta c}{gd^2} - k_1 + k_2\right]\frac{1}{2k_2}$$

or

$$\Delta d = \frac{d}{2k_2}\left[\frac{2c\Delta c}{gd} - k_1 + k_2\right]$$

This may be written

$$\Delta d = \frac{c\,\Delta c}{k_2 g} + \frac{d}{2}\left(1 - \frac{k_1}{k_2}\right)$$

From the geometry of the vector diagram:

$$\frac{c + \Delta c}{\tan(\beta + \Delta\theta)} = u \cos\beta$$

or

$$\Delta c = \frac{u\,\Delta\theta}{\cos\beta}$$

Now substituting this value for Δc and $u \sin\beta$ for c we have:

$$\Delta d = \frac{u^2 \sin\beta\,\Delta\theta}{k_2 g \cos\beta} + \frac{d}{2}\left(1 - \frac{k_1}{k_2}\right)$$

which may be simplified to:

$$\Delta d = \frac{u^2}{k_2 g} \tan\beta\,\Delta\theta + \frac{d}{2}\left(1 - \frac{k_1}{k_2}\right) \qquad (11)$$

Now let $k_1 - k_2 = \Delta k$ and we have

$$\Delta d = \frac{u^2}{kg} \tan\beta\,\Delta\theta - \frac{d}{2}\,\frac{\Delta k}{k}$$

or

$$\frac{\Delta d}{\Delta\theta} = \frac{u^2}{kg} \tan\beta - \frac{d}{2k}\,\frac{\Delta k}{\Delta\theta}$$

For large values of the Froude number one may consider

$$\tan\beta = \sin\beta$$

Then evaluating $\sin\beta$ from Equation(10) and equating it to $\tan\beta$ we have,

$$\frac{\Delta d}{\Delta\theta} = \frac{u^2}{kg}\sqrt{\frac{gd}{u^2}\left[k + \frac{d}{2\Delta d}\,\Delta k\right]} - \frac{d}{2k}\frac{\Delta k}{\Delta\theta}$$

But,

$$F = \frac{u^2}{gd}$$

therefore,

$$\frac{\Delta d}{\Delta\theta} = d\,\frac{\sqrt{F}}{k}\sqrt{k + \frac{\Delta k}{\Delta d}\frac{d}{2}} - \frac{d}{2k}\frac{\Delta k}{\Delta\theta}$$

Transpose the second term on the right and factor out $\Delta d/\Delta\theta$,

$$\frac{\Delta d}{\Delta\theta}\left[k + \frac{d}{2}\frac{\Delta k}{\Delta d}\right] = \sqrt{F}\,d\sqrt{k + \frac{\Delta k}{\Delta d}\frac{d}{2}}$$

then

$$\frac{\Delta d}{\Delta\theta} = \frac{\sqrt{F}\;d}{\sqrt{k + \frac{\Delta k}{\Delta d}\frac{d}{2}}}$$

Square both sides,

$$\left(\frac{\Delta d}{\Delta\theta}\right)^2 = \frac{F\,d^2}{k + \frac{\Delta k}{\Delta d}\frac{d}{2}}$$

and

$$\left(\frac{\Delta d}{\Delta\theta}\right)^2 k + \frac{d}{2}\frac{\Delta k}{\Delta\theta}\frac{\Delta d}{\Delta\theta} = Fd^2$$

Solving the quadratic in $\Delta d/\Delta\theta$ we have,

$$\frac{\Delta d}{\Delta\theta} = \frac{-\frac{d}{2}\frac{\Delta k}{\Delta\theta} + \sqrt{\frac{d^2}{4}\left(\frac{\Delta k}{\Delta\theta}\right)^2 + 4Fkd^2}}{2k}$$

or more simply

$$\frac{\Delta d}{\Delta\theta} = \frac{d}{4k}\left[-\frac{\Delta k}{\Delta\theta} + \sqrt{\left(\frac{\Delta k}{\Delta\theta}\right)^2 + 16kF}\right] \qquad (12)$$

6. Criterion for normal angular depth gradient

From Equation (15) which is

$$\frac{\frac{\Delta d}{\Delta \theta}}{\left(\frac{\Delta d}{\Delta \theta}\right)_n} = \frac{\frac{d}{4k}\left[-\frac{\Delta k}{\Delta \theta} + \sqrt{\left(\frac{\Delta k}{\Delta \theta}\right)^2 + 16kF}\,\right]}{d\sqrt{F}} \qquad (15)$$

we may obtain, upon equating $\frac{\Delta d}{\Delta \theta} \Big/ \left(\frac{\Delta d}{\Delta \theta}\right)_n$ to unity, the following,

$$-\frac{\Delta k}{\Delta \theta} + \sqrt{\left(\frac{\Delta k}{\Delta \theta}\right)^2 + 16kF} = 4k\sqrt{F}$$

which becomes, after transposing terms and squaring both sides,

$$\left(\frac{\Delta k}{\Delta \theta} + 4k\sqrt{F}\right)^2 = \left(\frac{\Delta k}{\Delta \theta}\right)^2 + 16kF$$

or

$$\left(\frac{\Delta k}{\Delta \theta}\right)^2 + 16k^2F + 8k\sqrt{F}\,\frac{\Delta k}{\Delta \theta} = \left(\frac{\Delta k}{\Delta \theta}\right)^2 + 16kF$$

Collecting terms and simplifying we obtain,

$$\frac{\Delta k}{\Delta \theta}\,\frac{1}{1-k} = 2\sqrt{F}$$

(b) Plottings of Experimental Data

Plottings of the type shown in Figs. 18-22 have been made, in a form suitable for reproduction, of the data for all the runs deemed usable. These drawings are on file in the office of the department of mechanics and hydraulics of the State University of Iowa with the master copy of the manuscript.